Nonlinear H_∞ Control of Robotic Manipulator

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Abstract

H_∞ control theory for nonlinear systems has been developed, which is based on the concept of the energy dissipation. A nonlinear H_∞ controller using the energy dissipation is designed in the sense of L_2-gain attenuation from a disturbance to performance and it is essential to find the solution of the Hamilton Jacobi (HJ) equation (or inequality) for application. However, it is difficult to obtain its solution in general. In this paper, the robot dynamics is transformed to a affine form to express the HJ inequality as a more tractable form, i.e., a nonlinear matrix inequality (NLMI), and its approximated solution is obtained from the fact that the terms in matrices which describe robot manipulator can be bounded.

1 Introduction

H_∞ controllers in linear systems can be obtained in the state space by solving Riccati equation [1], [2]. Another approach to H_∞ control in linear systems is a Linear Matrix Inequality (LMI) technique, where the solution is obtained by efficient convex optimization algorithms [10].

Recently H_∞ control problem for nonlinear systems has attracted attention of many researchers. Although the H_∞ control theory in nonlinear systems has been derived by the L_2-gain analysis based on the concept of the energy dissipation [3], [5], its applications are not easy due to the solvability of the Hamilton Jacobi (HJ) equation or inequality. The HJ equation (or inequality) is a first-order partial differential equation (or inequality) and it is difficult to obtain its solution in general. H_∞ control problem in nonlinear systems reduces to the existence of solution of HJ equation (or inequality). Van der Schaft suggested the successive approximated solution of the HJ equation [3]. Based on this method, Hu designed a nonlinear H_∞ controller for the inverted pendulum system [7]. Using modified Lyapunov function including a mixed term in link positions and velocities, Astolfi designed a robust PD controller for robot manipulator, whose gain was obtained by solving the associated HJ inequality [4]. As another application to robot manipulator, there was Hamiltonian optimization method using the fact that the approximated solution of HJ equation at equilibrium point is equal to the solution of the Riccati equation [6].

In this paper, the robot dynamics is transformed to the affine nonlinear system about states and input and the associated HJ inequality is derived in the form of a more tractable NLMI. The approximated solution of the NLMI can be obtained from the fact that the terms in matrices which describe the robot manipulator are bounded by trigonometric functions. If the matrices forming the NLMI is bounded, then we only need to solve the finite number of LMIs. For a two-degree-of-freedom planar manipulator with mass uncertainty, a H_∞ controller is designed and the robustness is shown through simulation.

This paper is organized as follow. In section II, we review the concept of the energy dissipation and an important theorem concerning the nonlinear H_∞ control problem, which are used in section III. In section III, the nonlinear system is represented in the form of affine nonlinear system about the state and inputs and the associated HJ inequality is transformed to the more tractable NLMI. In section IV, the dynamics of the robot manipulator is transformed to the affine form using modified error vector for tracking and a possible method is proposed to obtain the approximated solution to the NLMI. In section V, simulations are performed to confirm the robust performances of the proposed controller for robot manipulator under parameter uncertainty. In section VI, the conclusion is presented.

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2 Nonlinear State Feedback H_∞ Control

2.1 Energy Dissipative System

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x,w) \\ z &= h(x,w), \end{aligned} \tag{1}$$

where x is the state, w and z are the input and the output. With $\gamma > 0$, the system is said to be γ-dissipative if there exists a nonnegative energy storage function E with $E(x(0)) = 0$ such that for all w and T

$$\int_0^T \|z\|^2 dt - \gamma^2 \int_0^T \|w\|^2 dt \leq -E(x(T)). \tag{2}$$

When $\gamma = 1$, the inequality implies that the input energy is greater than or equal to the output energy, i.e., some energy is dissipated. The energy dissipation means that the L_2-gain of the system is less than or equal to γ. Obviously, the system is γ-dissipative if there exists a nonnegative function E such that the energy Hamiltonian function defined by

$$H = \|z\|^2 - \gamma^2\|w\|^2 + \frac{dE}{dt} \tag{3}$$

is nonpositive for all x and w in the domain of interest.

2.2 Nonlinear H_∞ Control Problem

Consider a nonlinear system expressed by

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h(x) + d(x)u, \quad h^T d = 0, \ d^T d > 0, \end{aligned} \tag{4}$$

where $x \in R^n$, $u \in R^m$, and $w \in R^w$ are the state, the control input, and disturbances, respectively. $z \in R^z$ represents the performance of the system.

To find a nonlinear state-feedback H_∞ control is to find a stabilizing state-feedback control input such that the closed-loop system has a L_2-gain equal to or less than γ in the input-to-output sense. This problem can be solved from the concept of energy dissipation.

A theorem concerning the solution of nonlinear H_∞ control problem described above is introduced without its proof in following theorem [3].

Theorem 1 *Given $\gamma > 0$, suppose there exists a C^1 positive definite function $E(x)$ with $E(0) = 0$ satisfying HJ inequality*

$$E_x f - \frac{1}{2}E_x\left(g_2\left[d^T d\right]^{-1} g_2^T - \frac{1}{\gamma^2}g_1^T g_1\right)E_x^T + \frac{1}{2}h^T h \leq 0 \tag{5}$$

where $E_x = \partial E^T/\partial x$, then the system (4) has the L_2-gain $\leq \gamma$ as well as the closed-loop stability with control input of

$$u = -\left[d^T d\right]^{-1} g_2^T E_x^T. \tag{6}$$

The above theorem shows that to construct a nonlinear H_∞ controller, it is essential to find the solution of the associated HJ inequality derived from input-output energy dissipation. If a solution exists, then it will guarantee the stability as well as the disturbance attenuation in the L_2-gain sense. However, the HJ inequality is a first order partial differential inequality and, in general, it is difficult to find its solution.

3 State Feedback H_∞ Controller in Affine Nonlinear System

In this section, it is shown that the associated HJ inequality can be transformed to a NLMI if the nonlinear system is described in suitable form and the solution of NLMI can be obtained easily from the fact that the matrices forming it are bounded.

Suppose that the nonlinear system can be described in the form

$$\begin{aligned} \dot{x} &= F(x)x + G_1(x)w + G_2(x)u \\ z &= H(x)x + D(x)u, \quad H^T D = 0, \ D^T D > 0, \end{aligned} \tag{7}$$

where $F(x)$, $G_1(x)$, $G_2(x)$, $H(x)$ and $D(x)$ are C^0 matrix-valued functions of suitable dimensions. If the nonlinear system can be transformed to Eq. (7), the derived HJ inequality is more tractable than Eq. (5).

The design of H_∞ controller for the nonlinear system in the affine form is summarized in the following theorem.

Theorem 2 *Given $\gamma > 0$, suppose there exists a C^0 matrix-valued function P satisfying*

$$\begin{aligned} &P^T(x)F(x) + F^T(x)P(x) \\ &+ \frac{1}{\gamma^2}P^T(x)G_1(x)G_1^T(x)P(x) + H^T(x)H(x) \\ &- P^T(x)G_2(x)\left[D^T(x)D(x)\right]^{-1} \cdot G_2^T(x)P(x) \leq 0 \end{aligned} \tag{8}$$

and there exists a non-negative function $E(x) \geq 0$ such that $\partial E/\partial x = 2x^T P^T$. Then the control input satisfying L_2-gain $\leq \gamma$ is

$$u = -\left[D^T(x)D(x)\right]^{-1} G_2^T P(x)x. \tag{9}$$

Proof: Take $E(x)$ as defined in the statement, then

$$
\begin{aligned}
\dot{E} &= \frac{\partial E}{\partial x}\dot{x} \\
&= 2x^T P^T \left(Fx + G_1 w + G_2 u\right) \\
&= x^T \left(P^T F + F^T P\right) x + 2x^T P^T G_1 w + 2x^T P^T G_2 u \\
&= \gamma^2 \|w\|^2 - \|z\|^2 + x^T (P^T F + F^T P) x \\
&\quad + 2x^T P^T G_1 w + 2x^T P^T G_2 u - \gamma^2 \|w\|^2 + \|z\|^2 \\
&= \gamma^2 \|w\|^2 - \|z\|^2 + x^T \left(P^T F + F^T P\right) x \\
&\quad + \frac{1}{\gamma^2} x^T P^T G_1 G_1^T P x - \gamma^2 \left\| w - \frac{1}{\gamma^2} G_1^T P x \right\|^2 \\
&\quad + 2x^T P^T G_2 u + x^T H^T H x + u^T D^T D u \\
&\quad + 2x^T H^T D u \\
&= \gamma^2 \|w\|^2 - \|z\|^2 + x^T \left(P^T F + F^T P\right) x \\
&\quad + \frac{1}{\gamma^2} x^T P^T G_1 G_1^T P x - \gamma^2 \left\| w - \frac{1}{\gamma^2} G_1^T P x \right\|^2 \\
&\quad + 2x^T P^T G_2 u + x^T H^T H x + u^T D^T D u \\
&\quad \left(H^T D = 0\right).
\end{aligned}
$$

When matrix D is not a square matrix, nonsingular matrix R satisfying $D^T D = R^T R$ is defined. Then

$$
\begin{aligned}
\dot{E} &= \gamma^2 \|w\|^2 - \|z\|^2 + x^T \left(P^T F + F^T P\right) x + x^T H^T H x \\
&\quad + \frac{1}{\gamma^2} x^T P^T G_1 G_1^T P x - \gamma^2 \left\| w - \frac{1}{\gamma^2} G_1^T P x \right\|^2 \\
&\quad + 2x^T P^T G_2 R^{-1} R u + u^T R^T R u \\
&= \gamma^2 \|w\|^2 - \|z\|^2 - \gamma^2 \left\| w - \frac{1}{\gamma^2} G_1^T P x \right\|^2 \\
&\quad + x^T \{P^T F + F^T P + \frac{1}{\gamma^2} P^T G_1 G_1^T P + H^T H \\
&\quad - P^T G_2 (R^T R)^{-1} G_2^T P\} x + \left\| R u + R^{-T} G_2^T P x \right\|^2 \\
&\leq \gamma^2 \|w\|^2 - \|z\|^2 \qquad \text{(by Eq. (8) and (9))}.
\end{aligned}
$$

Integrating the inequality results in

$$
E\left(x(T)\right) - E\left(x(0)\right) \leq \gamma^2 \int_0^T \|w\|^2 dt - \int_0^T \|z\|^2 dt.
$$

Since $E\left(x(T)\right) \geq 0$ and $E\left(x(0)\right) = 0$, the closed-loop ssytem becomes dissipative. ∎

To obtain the solution to the Eq. (8) easily, it is transformed to a NLMI using the Schur complement as

$$
\begin{bmatrix} P^T F + F^T P + H^T H + \frac{1}{\gamma^2} P^T G_1 G_1^T P & P^T G_2 \\ G_2^T P & D^T D \end{bmatrix} \leq 0.
$$

Solving the above NLMI yields convex optimization problem. Unlike the linear case, this convex problem is not finite dimensional. However, if the matrices forming the NLMI are bounded, then we only need to solve a finite number of LMIs [8].

4 H_∞ Control for Robot Manipulator

4.1 Dynamic Equation in Affine Form

Consider the dynamics of a n-link robot manipulator of

$$
M(q)\ddot{q} + V\left(q, \dot{q}\right)\dot{q} + G(q) = \tau, \tag{10}
$$

where $q \in R^n$ is the joint position , $M(q) \in R^{n\times n}$ is the positive definite symmetric inertia matrix, $V\left(q, \dot{q}\right) \in R^{n\times n}$ represents the centripetal and coriolis torque, and $G(q) \in R^n$ represents the gravitational torque.

Modified error vector s is defined as

$$
\begin{aligned}
s &= \left(\dot{q} - \dot{q}_d\right) + \Lambda(q - q_d) \\
&= \dot{q} - \{\dot{q}_d - \Lambda(q - q_d)\},
\end{aligned}
$$

where q_d and $\dot{q}_d$ are the desired position and velocity, respectively.

Also, by defining $\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$,

$$
s = \dot{q} - \dot{q}_r. \tag{11}
$$

If the elements of vector s approach to zeros as $t \to \infty$, so does the tracking error of each joint.

Using the modified error vector, we can transform the robot dynamics to a affine form.

Proposition 1. Using Eq. (11) and a suitable choice of control input τ, Eq. (10) can be transformed to

$$
\dot{x} = F\left(q, \dot{q}\right)x + G_1(q)w + G_2(q)u \tag{12}
$$

where $x = (s_1, \cdots, s_n)$, $F\left(q, \dot{q}\right) = -M^{-1}(q)V\left(q, \dot{q}\right)$, $G_1 = M^{-1}(q)$, and $G_2 = M^{-1}(q)$.

Proof: To transform Eq. (10) to Eq. (12) the following control input is proposed.

$$
\tau = \hat{M}(q)\ddot{q}_r + \hat{V}\left(q, \dot{q}\right)\dot{q}_r + \hat{G}(q) + u, \tag{13}
$$

where $\hat{M}(q)$, $\hat{V}\left(q, \dot{q}\right)$ and $\hat{G}(q)$ are estimates of $M(q)$, $V\left(q, \dot{q}\right)$, and $G(q)$, respetively.

Substituting Eq. (13) into Eq. (10),

$$
M(q)\dot{s} + V\left(q, \dot{q}\right)s = \tilde{M}(q)\ddot{q}_r + \tilde{V}\left(q, \dot{q}\right)\dot{q}_r + \tilde{G}(q) + u. \tag{14}
$$

where the model estimate errors are

$$
\begin{aligned}
\tilde{M}(q) &= \hat{M}(q) - M(q), \\
\tilde{V}(q, \dot{q}) &= \hat{V}(q, \dot{q}) - V(q, \dot{q}), \\
\tilde{G}(q) &= \hat{G}(q) - G(q).
\end{aligned}
$$

Define a disturbance vector as

$$w = \tilde{M}(q)\ddot{q}_r + \tilde{V}(q,\dot{q})\dot{q}_r + \tilde{G}(q),$$

then Eq. (14) becomes

$$M(q)\dot{s} = -V(q,\dot{q})\,s + w + u. \quad (15)$$

Multiplying the inverse of $M(q)$ to both sides of Eq. (15), we can obtain Eq. (12). ∎

4.2 The Solution to HJ Inequality using LMI

To derive the HJ inequality for the robot manipulator dynamics transformed to affine form, each matrix term of Eq. (12) is substituted into Eq. (8). Then

$$\left(MP^{-T}\right)^{-1} V - V^T \left(P^{-1}M^T\right)^{-1} + H^T H + \frac{1}{\gamma^2}\left(MP^{-T}\right)^{-1}\left(P^{-1}M^{-T}\right)^{-1} - \left(MP^{-T}\right)^{-1}\left(D^T D\right)^{-1}\left(P^{-1}M^T\right) \leq 0.$$

Premultiplying and postmultiplying the inequality by positive definite matrices MP^{-T} and $P^{-1}M^T$ respectively, then the HJ inequality becomes

$$-VQM^T - MQ^T V^T + MQ^T H^T HQM^T + \frac{1}{\gamma^2}I - \left(D^T D\right)^{-1} \leq 0, \quad (16)$$

where $Q = P^{-1}$. Using the Schur complement, Eq. (16) can be described as a NLMI

$$\begin{bmatrix} -VQM^T - MQ^T V^T + \frac{1}{\gamma^2}I - (D^T D)^{-1} & MQ^T H^T \\ HQM^T & -I \end{bmatrix} \leq 0. \quad (17)$$

The matrices $M(q)$ and $V(q,\dot{q})$ is the nonlinear function of q and $\dot{q}$ in Eq. (17). However, those matrices include trigonometric functions and can be bounded when each joint velocity ranges between two determined extremal values. Using this fact, we suppose that the matrices forming the above NLMI vary in some bounded sets of the space of matrices, i.e.,

$$[M(q), V(q,\dot{q}), H, D] \in C_0\{[M_i, V_i, H, D]|_{i\in\{1,2,\cdots,L\}}\},$$

where C_0 denotes the convex hull.

Therefore, if

$$\begin{bmatrix} -V_i QM_i^T - M_i Q^T V_i^T + \frac{I}{\gamma^2} - (D^T D)^{-1} & M_i Q^T H^T \\ HQM_i^T & -I \end{bmatrix} \leq 0$$

have a common nonsingular matrix solution Q for all $i \in \{1,2,\cdots,L\}$, then Q is also a solution to Eq. (17) and the stabilizing control input is determined as

$$u = -\left(D^T D\right)^{-1} G_2^T Q^{-1} s.$$

This approach provides a tractable method to get constant solutions to NLMI, which can be used to design the control input. However, this approach generally leads to conservative results if the prescribed bound is large. This can be seen in simulation results.

5 Simulation

A nonlinear H_∞ controller is designed for two-degree-of-freedom planar robot manipulator with uncertainty in its mass. Simulations were performed to evaluate the proposed H_∞ controller. The objective of the simulation is to show the enhancement of robustness to parameter uncertainty. The system model matrices forming LMIs are determined by the bound of parameter uncertainty and the property of trigonometric functions. The set of dynamic parameter is summarized in Table 5. Simulations are performed for two cases according to the size of mass bound. As an extreme disturbance, the mass of link 2 is assumed to vary by 50% at 2 second. The LMIs for the control gain matrix are solved using an efficient convex optimization algorithm in Matlab toolbox. It should be noted that the easiness of controller tuning can be obtained since the solution of LMIs, if any, is found easily by an optimization algorithm.

The joints of manipulator are commanded to trace trajectories shown in Fig. 1 with some initial errors. The initial errors of the joints are 17.19° and 22.91°, respectively. The estimates of the manipulator model matrices in Eq. (13) are assumed to be $\hat{M} = 0$ and $\hat{V} = 0$. The estimate of the gravity torque G is determined from the equation in the dynamics using the estimates of mass $\hat{m}_1$=1.8 and $\hat{m}_2$=0.8.

The performance level can be determined by parameter γ and weighting matrix H and the control input energy can be adjusted by using matrix D. In the simulation, matrix H and matrix D are selected as

$$H = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix}^T, \quad D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^T.$$

The value of γ is selected as 7 such that the solution of given LMIs is feasible.

The position error and torque are shown in Fig. 2 ∼ Fig. 5, respectively. The error in each case is similar. However, when the bound of mass uncertainty is large,

Table 1: Manipulator parameters used in the simulation

	Real Length	Real Mass	Bound of Mass	
			Case 1	Case 2
Link1	0.5 m	2 kg	[1.5,2.5]	[0.5,1.5]
Link2	0.3 m	1kg	[0.5,4]	[0.2,3]

the control torque increases in initial state. This shows that the large prescribed bound leads to a conservative result.

As long as the described control input is within feasible range, the proposed controller shows satisfactory robustness performance even under the large parameter uncertainty.

6 Conclusion

We proposed a robust controller for a tracking and disturbance attenuation of robotic manipulator. The error of parameter estimates is considered as a disturbance and the robustness to model uncertainty is achieved in the sense of L_2-gain attenuation from the disturbance to performance measure. The associated HJ inequality is transformed to NLMI and its approximated solution is obtained from the fact that the terms in matrices which describe robot manipulator can be bounded by trigonometric functions. The application of the proposed controller is simple since the control gain matrix can be obtained easily by an efficient convex optimization algorithm. The proposed controller is applied to a two-degree-of freedom planar manipulator and its computer simulation shows that the controller sustains its performance under the uncertainty of mass.

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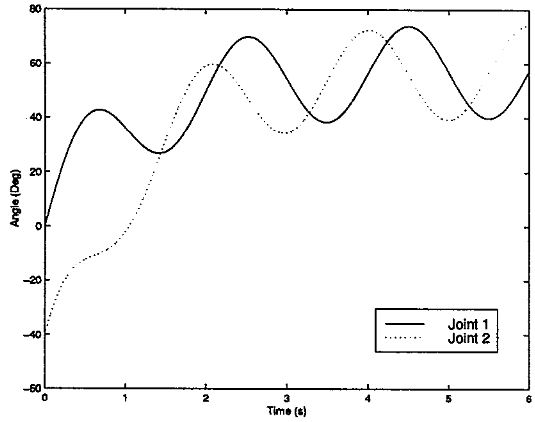

Figure 1: Desired trajectory of joints.

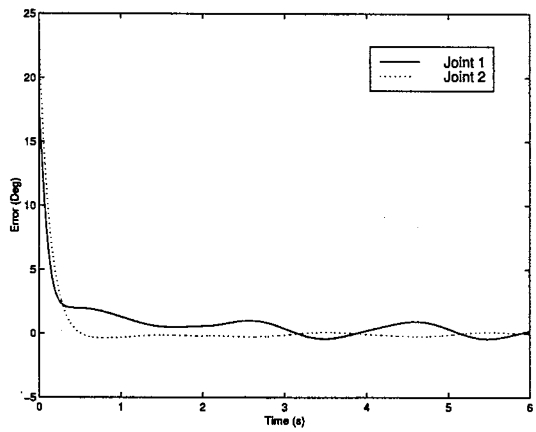

Figure 2: Position error in Case 1.

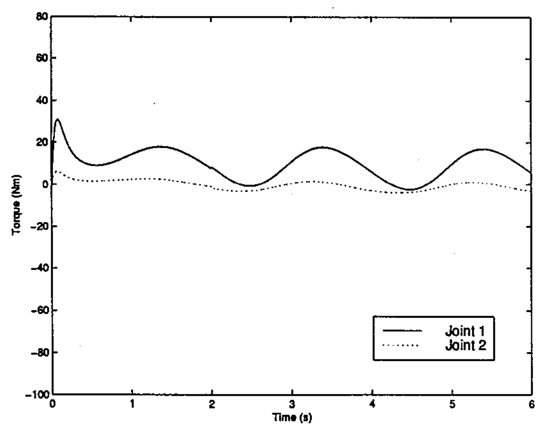

Figure 3: Torque in Case 1.

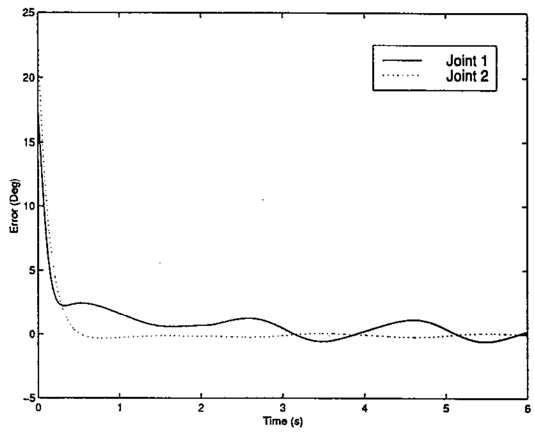

Figure 4: Position error in Case 2.

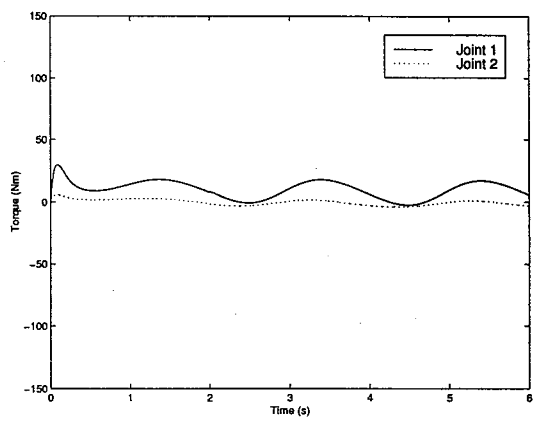

Figure 5: Torque in Case 2.

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