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Stabilization of systems with changing dynamics

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Abstract

We present a framework for designing stable control schemes for systems with changing dynamics – systems whose dynamics change as the state evolves through different regions in the state space. Systems with changing dynamics (SCD) form a subset of hybrid systems; their stabilization is therefore a problem in hybrid control. It is often difficult or even impossible to design a single controller that would stabilize a SCD. An appealing alternative are switching control schemes, where a different controller is employed in each dynamic regime and the stability of the overall system is ensured through an appropriate switching scheme. We formulate a set of sufficient conditions for the stability of a switching control scheme. We show that by imposing a hierarchy among the controllers, sufficient conditions can be formulated in a form suitable for the controller design. The hierarchy is formally defined through a partial order. With partial order, the study of the stability of the system is reduced to the study of relationships among the immediate neighbors in the partial order. This significantly simplifies the analysis and design processes. The methodology is applied to stabilization (of a relative equilibrium) of a two-wheel mobile robot of the Hilare type, where the wheels are allowed to slip. The example demonstrates that the approach is easy to use and that the partial order concept naturally leads to modularity in the controller design.

Keywords: systems with changing dynamics, hybrid systems, switching controllers, partial order, Hilare robot

I. Introduction

Design of controllers for hybrid systems remains a challenging problem and is the subject of considerable research. A common assumption of many existing design methodologies is that a physical plant with continuous dynamics is controlled by a supervisor in the form of a finite automaton. Our work addresses a different problem. We study systems whose dynamics change in different regions of the state space. There is no natural splitting between the continuous and discrete layers of the system behavior, the discrete behavior arises completely from the continuous. Because of the inherently discontinuous dynamics, the control task becomes in many respects more challenging. The motivation for our work comes from problems that arise in practice and have features not handled by prior techniques. In particular, we are interested in applications in locomotion and robotics.

In this paper, we study stabilization of systems with changing dynamics (SCD). The goal of control is to bring the system into a desired dynamic regime and stabilize an equilibrium set in that regime. The equilibrium set might be for example a relative equilibrium or a limit cycle. To reach the desired dynamic regime, the system must pass through other regimes, and it is possible that they have to be traversed in a certain order for the system to be stable. Because of discontinuous dynamics, it is difficult to design a single controller that would stabilize a SCD. To reduce the complexity of the problem we propose an alternative strategy, whereby a different controller is employed in each dynamic regime; as the system evolves, we switch among different controllers. One of the goals of this paper is to investigate what conditions must be satisfied by such a switching controller to guarantee the stability of the system. A second goal is to suggest a paradigm for designing a controller that satisfies such conditions. We observe that in many tasks the stability in the sense of Lyapunov is too restrictive, so we concentrate on the control schemes that only guarantee the convergence of the trajectories to the desired set. However, our basic methodology does not preclude Lyapunov stability.

A. Relation to previous work

A starting point for the controller design is a choice of formalism for describing hybrid systems. Several alternatives can be found in the literature. Building upon their work on the automata theory, [1] and [2] defined the notion of a hybrid automaton. Theory of dynamical systems is the basis of the model in [3]. Examples of other works in this category are [4] and [5]. References to several other models can be found in [6]. We use models in this second group for our work.

Majority of works on hybrid controller design rely on certain properties of the system. An early work is [7], where piecewise-linear systems are used as underlying model for hybrid systems. Algorithms for automated design of controllers for a simplified version of hybrid automata are described in [8] and [9]. A methodology for controlling multiple cooperating agents is proposed in [10]. A game-theoretic framework for designing hybrid controllers was proposed in [11] and applied to intelligent highway systems [12] and air traffic control systems [13]. In [14], timed Petri nets are used to model hybrid systems; supervisory control framework is employed for their control. A hybrid controller for the so called cascade systems was proposed in [15]. Optimal control for hybrid systems was explored in [16]; it was applied to trajectory synthesis in [17]. A controllability test and a planning method for a class of hybrid systems called stratified systems was developed in [18].

A number of authors considered stability of hybrid systems. Classical Lyapunov theory has been extended for non-smooth and hybrid systems in [19], [20]. Multiple Lyapunov functions were proposed for stability analysis of hybrid systems in [21], [22], [23]. A controller design methodology based on multiple Lyapunov functions is described in [24]. An important contribution towards the application of multiple Lyapunov functions for practical controller design is the work in [25] and [26]. There, the problem of finding a set of Lyapunov functions for piecewise-linear systems is transformed into a (numerically tractable) problem of solving a system of linear matrix inequalities. In [27], this ideas were used to derive a simplified test for stability of a hybrid system modeled with a Petri net. The idea of guiding the system through a sequence of equilibrium points in order to stabilize it was employed in [28]. An assumption that is common to most of these works (an exception is [28]) is that every subsystem has the same equilibrium point which has to be stabilized. However, hybrid systems can exhibit much richer behavior: the system might switch between multiple equilibrium sets before reaching the final state. Another common assumption is that the switches between the controllers are either explicitly controlled, or that the switching surfaces can be explicitly characterized. These assumptions are quite restrictive and one of our goals is to overcome these limitations.

B. Our approach

As in [21], [22], [23], multiple Lyapunov functions are also the basis for the stability analysis in the present work. However, we assume that the equilibrium sets among different dynamic regimes are different. The convergence to the desired equilibrium set is achieved not only by switching among different controllers, but also by switching among different equilibrium sets. We show that a desired behavior of the system can be obtained by prescribing a hierarchy among the controllers. The hierarchy is formally defined through a partial order relation. If the switches between the controllers are consistent with the chosen partial order, the convergence of the system towards the desired equilibrium set follows. We also propose a methodology for designing a set of controllers that satisfy such requirements. An important feature of our work is that the convergence of the system to the equilibrium set can be guaranteed by only satisfying certain relations between the controllers "locally" (with respect to the partial order). This leads to modularity of the design process and considerably simplifies the synthesis problem.

C. Motivating example

This section briefly describes an admittedly simple motivating example. A realistic application of this work is presented in Section V. Consider a piecewise linear system whose dynamics are described by:

$$\begin{aligned} M_1 : \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & & \text{if } \|x\| \leq 1 \\ M_2 : \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & & \text{if } 1 < \|x\| \leq 3 \\ M_3 : \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\left(\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 5 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) & & \text{otherwise} \end{aligned} \tag{1}$$

The partition of the state space into three dynamic regimes is shown in Figure 1.a.

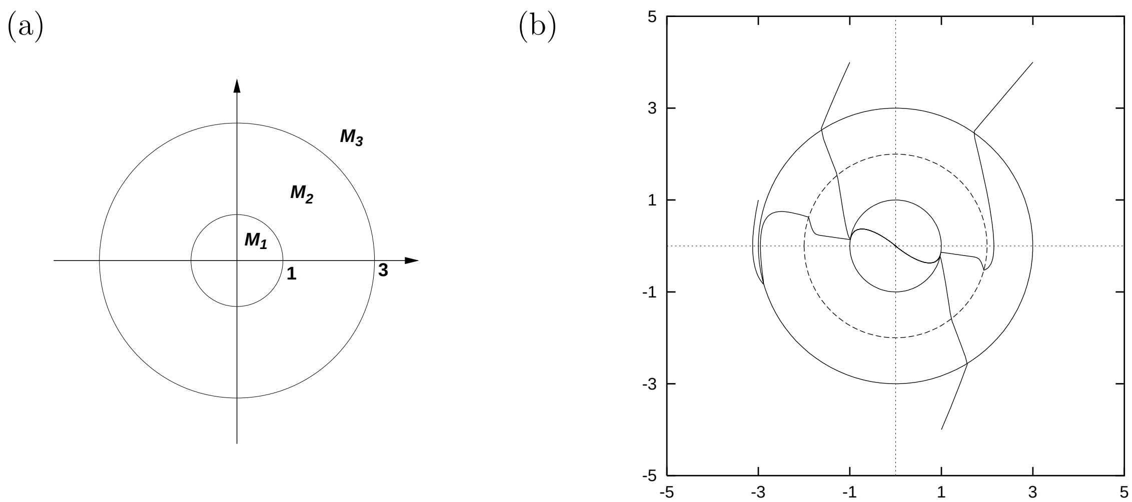

Fig. 1. (a) Partition of the state space; (b) some sample trajectories.

The aim of the control is to make the origin globally asymptotically stable. Note that the dynamics in the regime M_3 is the negative of the dynamics in M_1, modulo a bias term that makes the equilibrium point in M_3 different from the origin. Therefore, if a linear controller:

$$u = g(x_1, x_2) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

stabilizes regime M_1, the same controller will necessarily destabilize M_3. The only way to globally stabilize the origin is thus to design different controllers for M_1 and M_3.

Let:

$$K_1 = \left[\begin{array}{cc} -\frac{4}{3} & -7 \end{array} \right] \tag{3}$$

be the gains for the controller g_1 for M_1 and

$$K_3 = \left[\begin{array}{cc} -\frac{4}{3} & -3 \end{array} \right] \tag{4}$$

the gains for the controller g_3 for M_3. A short calculation shows that g_1 makes both poles for M_1 equal to $p_1 = p_2 = -1$, while g_3 achieves the same for M_3. In addition, g_1 and g_3 both stabilize M_2: with the controller g_1, the poles of the closed loop system are $p_1 = -3$ and $p_2 = -\frac{1}{3}$, while with the controller g_3, the poles are $p_1 = \frac{-11-4\sqrt{7}}{3}$ and $p_2 = \frac{-11+4\sqrt{7}}{3}$. It is also worth noting that g_3 moves the equilibrium point of M_3 to $(-\frac{1}{3}, 0)$.

Now define the following controller:

$$g(x) = \begin{cases} g_1(x) & \text{if } \|x\| \leq 2 \\ g_3(x) & \text{otherwise} \end{cases} \tag{5}$$

It is not difficult to show that with g defined as above, the function $V = \|x\|^2$ is nondecreasing in the regimes M_1 and M_2. This implies that once the system enters the domain of M_2, it does not switch back to M_3. And since the equilibrium point of M_3 is $(-\frac{1}{3}, 0)$, all the trajectories of M_3 will eventually intersect the domain of M_2 and therefore converge to the origin. Some trajectories of the system are shown in Figure 1.b.

We also observe that the defined control scheme is robust with respect to the switches. The switch between g_1 and g_3 occurs within the domain of M_2 where both controllers are stable. This implies that the system will behave properly even if the state is not known precisely and the switch does not occur exactly on the set $\|x\| = 2$.

The example is not physically motivated and was constructed in order to motivate our approach. However, it shows how our work differs from those above. In [19]-[24], it is assumed that all the sub-systems have the same equilibrium point. Therefore, none of those results can be applied here. Furthermore, while all the subsystems are linear, the domains of each of the subsystems are not polygonal and convex, which makes the application of the results in [25]-[27] difficult. Also, the switches between different dynamic regimes are autonomous, so the scheme from [28] can not be used.

D. Contents of the paper

The paper is organized as follows. We describe our formal model for a SCD in Section II. In Section III we review some notions for stability analysis on manifolds and then formulate three propositions that give sufficient conditions for the stability of a switching control scheme. The propositions are progressively less abstract and lead to a practical methodology for the controller synthesis. Some techniques for designing switching controllers that satisfy sufficient conditions for stability are described in Section IV. In Section V we apply the methodology to the problem of stabilization of a Hilare robot whose wheels can slip. Performance of the controller is illustrated with simulation results. We conclude the paper with a discussion.

II. Modeling

In this section we describe the basic model of a dynamical system that will be used in the paper. The main feature of a SCD is that as the system moves from one region of the state space into another, dynamic equations of the system change. Partition of the state space into different regions will be described by a collection of manifolds. Each of the manifolds will be called a discrete state. On each of the manifolds (in each discrete state), the system dynamics can be described by a (controlled) vector field. Since a vector field describes the evolution of the continuous state, these control inputs will be called continuous. On the other hand, in some cases we will be able to force the system to switch from one regime to a different regime. Such a discrete selection will be described by a set of discrete controls. Finally, we will need a map that describes how the system switches between dynamic regimes. In other word, we need a function that maps the continuous state in the current dynamic regime (discrete state), and current continuous and discrete controls, into a new dynamic regime. We thus arrive at the following formal definition:

Definition II.1: A *system with changing dynamics (SCD)* is a tuple:

$$\text{SCD} = (\Xi, \mathcal{M}, \mathcal{U}, \Gamma, \mathcal{F}, \Sigma) \tag{6}$$

where
1. $\Xi \subset \mathbb{Z}$ is a (finite) set of discrete states.
2. $\mathcal{M} = \{M_i\}_{i\in\Xi}$ is a collection of (differentiable, connected) manifolds. For simplicity, we assume $M_i \subseteq \mathbb{R}^n$ for some n.
3. $\mathcal{U} \subset \mathbb{R}^m$ is the set of continuous inputs.
4. $\Gamma \subset \mathbb{Z}$ is the set of discrete inputs.
5. $\mathcal{F} = \{f_i\}_{i\in\Xi}$ is a set of ($\mathcal{C}^1$) vector fields:

$$\begin{aligned} f_i : \quad & M_i \times \mathcal{U} \to TM_i \\ f_i : \quad & (x,u) \mapsto f_i(x,u) \in T_x M_i \end{aligned}$$

6. $\Sigma : \Xi \times \cup_{i\in\Xi} M_i \times \Gamma \times \mathcal{U} \to \Xi$ is a function describing the discrete evolution of the system.

The evolution of a SCD is given in the following way. The system evolves on M_i following the vector field f_i as long as $\Sigma(i,x,\eta,u) = i$. When $\Sigma(i,x,\eta,u)$ becomes equal to $j \neq i$, the system dynamics switches to (M_j, f_j). The value of $\Sigma(i,x,\eta,u)$ can change either because the trajectory of the system leaves the manifold M_i and enters M_j, or because the discrete input η changes. We will assume that as the dynamics of the system changes, the continuous state remains the same. In other words, the evolution of the continuous state x will be continuous[1]. This implies that the following condition must hold:

$$\Sigma(x,.,.,.) = i \quad \Rightarrow \quad x \in M_i \tag{7}$$

In general, the vector fields in $\mathcal{F}$ will be different, reflecting changes in the dynamics of the system. Also the dimensions of the manifolds in $\mathcal{M}$ might be different, as shown for example in Fig. 2.a. The system in the figure evolves on four manifolds: M_1, M_2 and M_3 of dimension 2 and M_4 of dimension 1. The target equilibrium set $E_4 \subset M_4$ consists of

[1]Most of the results in the paper can be generalized to systems with jumps.

a single point. Note that the manifolds are not disjoint and that M_4 is a subset of M_1, M_2 and M_3. It often helps to represent the topology of a SCD with a graph. The vertices of the graph correspond to the elements of $\mathcal{M}$ (dynamic regimes). There will be an edge from a vertex associated with M_i to a vertex associated with M_j if it is possible to switch from M_i to M_j. In other words, there exist x, η, and u, such that $\Sigma(i, x, \eta, u) = j$. For example, if we assume that a nonempty intersection of two manifolds implies that it is possible to switch between them, the graph for the system in Fig. 2.a is Fig. 2.b.

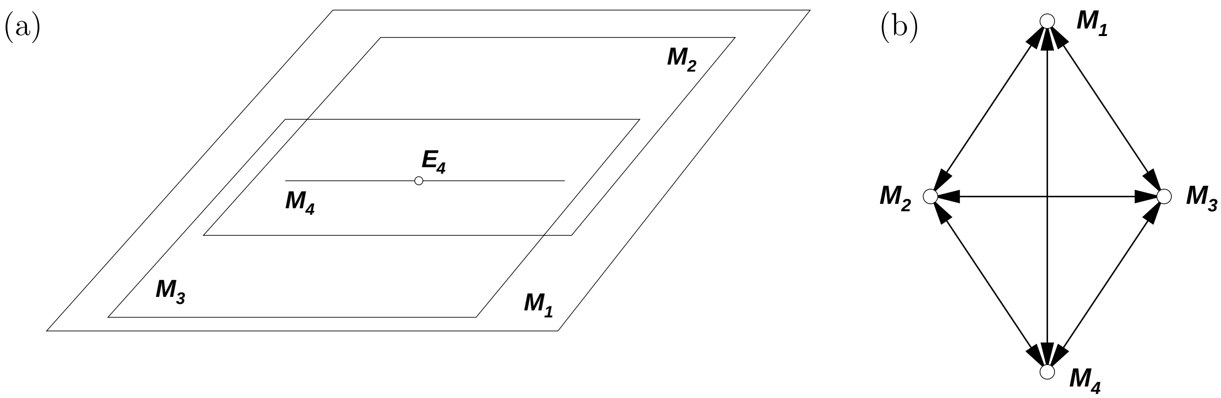

Fig. 2. (a) A sequence of embedded manifolds; (b) the corresponding graph.

An important issue for a SCD is the existence and uniqueness of solutions. Assuming that switches in the dynamic behavior are deterministic and that the vector fields in $\mathcal{F}$ are Lipschitz continuous, a unique solution is guaranteed to exist on each M_i. However, the overall evolution of the system might not be unique if chattering occurs. One way to avoid such anomalies is to introduce hysteresis in the switching rules. We will not pursue these issues further and we refer the reader to [29]. In the rest of the paper we assume that no chattering occurs so that a unique solution for system dynamics exists.

A. *Control Strategy*

In this paper we study stabilization of SCD's. The control task is to stabilize a submanifold E_n in a particular dynamic regime, $E_n \subseteq M_n$. Depending on the application, it might be necessary to achieve asymptotic stability or maybe only convergence of the trajectories of the system to E_n. In both cases, the control task is complicated by the fact that it is not known in advance what manifolds the dynamical system will traverse. In particular, it is possible that the system switches autonomously between different manifolds. It is also clear that switching might be unpredictable due to external disturbances.

Let $\Omega = (\Xi, \mathcal{M}, \mathcal{U}, \Gamma, \mathcal{F}, \Sigma)$ be a SCD. A natural way to control a SCD is to design a controller for each of the dynamic regimes. Therefore, for each manifold $M_i \in \mathcal{M}$ we design a controller g_i:

$$g_i : M_i \rightarrow \mathcal{U} \tag{8}$$

The function g_i will be assumed to be measurable, but it can be discontinuous. The evolution of the discrete state also depends on the discrete input η. We must therefore also design a discrete controller:

$$\mathcal{S} : \Xi \times \cup_{i \in \Xi} M_i \rightarrow \Gamma \tag{9}$$

which at each state $(i, x) \in \Xi \times \cup_{i\in\Xi} M_i$ selects a discrete input. consider the set of all discrete states reachable from the current combined (discrete and continuous) state, $D(i, x) = \{\Sigma(i, x, \eta, g_i(x)) \mid \eta \in \Gamma\}$. Clearly, by choosing an appropriate discrete input, we can force the system to switch to one of the discrete states in $D(i, x)$. Without loss of generality we can assume that $\Gamma = \Xi$ and that whenever $j \in D(i, x)$, $\Sigma(i, x, j, g_i(x)) = j$. For this reason, $\mathcal{S}$ will be also called a *switching function*. The collection of controllers $\mathcal{G} = \{g_i\}_{i\in\Xi}$ and the switching function (discrete controller) $\mathcal{S}$ form a *switching controller*, $(\mathcal{G}, \mathcal{S})$. Note that the switching controller turns Ω into an autonomous system, with the vector fields in $\mathcal{F}$ depending only on the continuous state x and the function Σ depending on the combined state (i, x).

When defining a SCD , we did not put any restrictions on $\mathcal{M}$ and $\mathcal{F}$. In particular, the manifolds in $\mathcal{M}$ or the vector fields in $\mathcal{F}$ need not be different. In this way, the case when several different controllers are used in the same dynamic regime can be treated in the same framework. If we design k different controllers for the dynamic regime (M_i, f_i), we can describe such a system by introducing additional elements $M_i^1, \dots, M_i^k$ to $\mathcal{M}$, and $f_i^1, \dots, f_i^k$ to $\mathcal{F}$, where $M_i^1 = \dots = M_i^k = M_i$ and $f_i^j(x) = f_i(x, g_j(x))$. In this way, we can always assume that exactly one controller is defined in each dynamic regime.

III. Stability of switching controllers

A. Stability theory on manifolds

We are interested in stabilizing submanifolds (possibly unbounded). The conventional Lyapunov theory can not be directly applied in this setting so we need some additional definitions (see [30]).

Definition III.1: A *distance* between a point x and a set $E \subseteq \mathbb{R}^n$ is defined by:

$$\rho(x, E) = \inf_{y\in E} d(x, y) \tag{10}$$

A *ball* with radius R around E is the set:

$$B(E, R) = \{x \mid \rho(x, E) < R\}$$

Definition III.2: A smooth manifold $E \subset M$ is *locally stable* if for any $R > 0$ there exist $r > 0$ such that if $\rho(x(t_0), E) < r$ then $\rho(x(t), E) < R$ for every $t > t_0$. It is *locally attractive* if there exists $R > 0$ such that if $\rho(x(t_0), E) < R$ then $\lim_{t\to\infty} \rho(x(t), E) = 0$. It is *locally asymptotically stable* if it is both locally stable and locally attractive.

Theorem III.3 ([31], [32]) If for a control system Ω there exists a C^1 function $V : M \to \mathbb{R}$, such that:
(1) $V(x) \geq 0$ and $V(x) = 0 \Leftrightarrow x \in E$;
(2) there exists a monotonically increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, $\alpha(0) = 0$, such that $\alpha(\rho(x, E)) < V(x)$;
(3) there exists a monotonically increasing function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, $\beta(0) = 0$, such that $V(x) < \beta(\rho(x, E))$;
(4) $\dot{V}(x) \leq 0$, where $\dot{V}$ is the derivative of V along the trajectories of Ω;
then the manifold E is locally stable. If in addition:
(5) there exists a monotonically increasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(0) = 0$, such that $\dot{V}(x) \leq -\gamma(\rho(x, E)) < 0$,
then E is locally asymptotically stable.

B. Sufficient conditions for convergence

Take a SCD $\Omega = (\Xi, \mathcal{M}, \mathcal{U}, \Gamma, \mathcal{F}, \Sigma)$ and a switching controller $(\mathcal{G}, \mathcal{S})$. Assume the control task is to stabilize $E_n \subseteq M_n$ and let the controller g_n stabilize E_n. Assume we can construct a Lyapunov function V_n which satisfies the conditions (1)-(5) of Theorem III.3. The following proposition gives sufficient conditions for E_n to be globally attractive:

Proposition III.4: Let the switching scheme $\mathcal{S}$ satisfy the following conditions:
(1.1) There exists $L > 0$ such that $\mathcal{S}(n, x) = n$ for every $x \in B(E_n, L) \cap M_n$.
(1.2) For any trajectory $x(t)$, there exists a $\Delta > 0$ and an infinite sequence $\{t_i\}$ whose elements satisfy:
(a) for every $t \in [t_i, t_i + \Delta]$, $\mathcal{S}(\eta(t), x(t)) = n$;
(b) $V_n(t_i + \Delta) \geq V_n(t_{i+1})$.

Then the submanifold E_n is globally attractive.

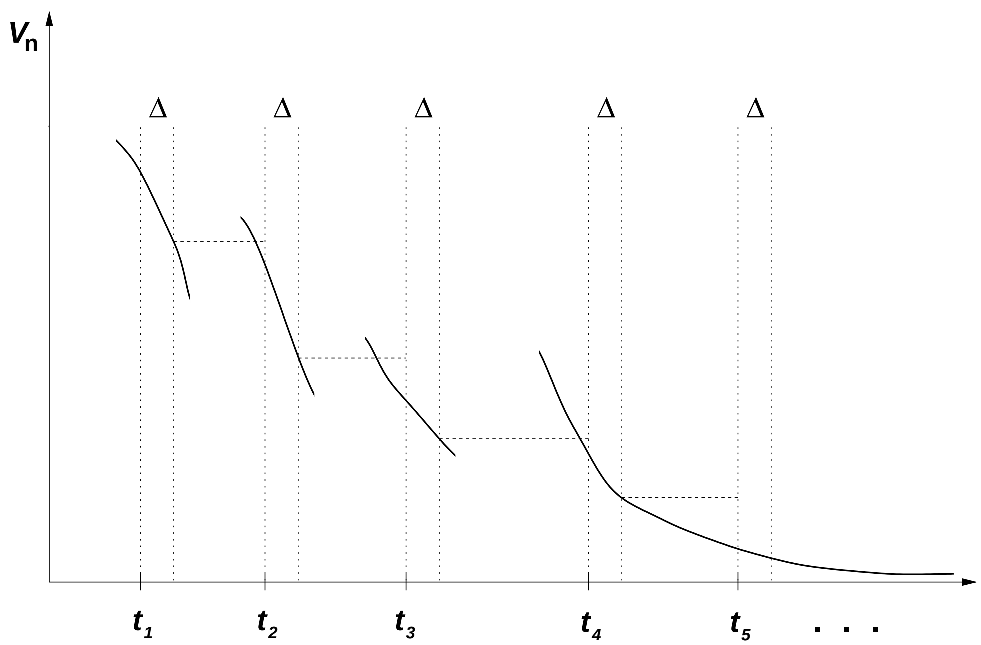

Fig. 3. Values of the Lyapunov function and a sequence satisfying the condition (1.2b) of Proposition III.4. The function is only shown when $\mathcal{S}(\eta(t), x(t)) = n$.

Remark III.5: The condition (1.1) guarantees that there is a region around E_n in which it is not possible to switch from g_n to some controller g_i, $i \neq n$. The condition (1.2) states that regardless of the current state, the system trajectory will eventually enter the manifold M_n and stay on it for at least Δ. Furthermore, we can find a sequence of subintervals of length at least Δ so that the Lyapunov function restricted to the union of these intervals is monotonically decreasing (see Fig. 3).

Proof: Let $\{t_k\}$ be a sequence given by the condition (1.2). Since the Lyapunov function V_n is monotonically decreasing when the system evolves on M_n, the condition (1.2b) implies that $t_{i+1} - t_i \geq \Delta$. Now take $I_n = \cup_{k \in \mathbb{N}} [t_k, t_k + \Delta]$ and consider the system evolving on I_n. By assumption, V_n satisfies the conditions of Theorem III.3, so we can find monotonically increasing functions α, β and γ such that $\alpha(\rho(x, E_n)) < V_n(x) < \beta(\rho(x, E_n))$ and $\dot{V}_n(x) \leq -\gamma(\rho(x, E_n)) < 0$. Let $r = \rho(x(0), E_n)$ and let ϵ be an arbitrary number such that $0 < \epsilon < r$. Then we can find $\delta > 0$ such that $\beta(\delta) < \alpha(\epsilon)$. Let K be an integer such that $K > \frac{\beta(r)}{\Delta\gamma(\delta)}$ and take $\tau = t_K + \Delta$. Suppose that $\rho(x(t), E_n) > \epsilon$ for every $t \in I_n \cap [0, \tau]$. Then we have:

$$\begin{aligned} 0 &< \alpha(\epsilon) \le V_n(x(\tau)) = V_n(x(t_K)) + \int_{t_K}^{t_K+\Delta} \dot{V}_n(x(t))\, dt \\ &\le V_n(x(t_K)) - \int_{t_K}^{t_K+\Delta} \gamma(\rho(x(t), E_n))\, dt \le V_n(x(t_K)) - \int_{t_K}^{t_K+\Delta} \gamma(\delta)\, dt \\ &= V_n(x(t_K)) - \Delta\gamma(\delta) \le V_n(x(t_{K-1}) + \Delta) - \Delta\gamma(\delta) \le \ldots \\ &\le V_n(x(0)) - K\Delta\gamma(\delta) \le \beta(r) - K\Delta\gamma(\delta) < 0 \end{aligned} \tag{11}$$

This is a contradiction, implying that there exists $\tau' \in I_n \cap [0, \tau]$ such that $\rho(x(\tau'), E_n) < \delta$. But then for every $t \in I_n$ such that $t > \tau'$:

$$\alpha(\rho(x(t), E_n)) \le V_n(x(t)) \le V_n(x(\tau')) \le \beta(\delta) < \alpha(\epsilon)$$

which implies:

$$\rho(x(t), E_n) < \epsilon \;\; \forall t > \tau', t \in I_n$$

This shows that $\rho(x(t), E_n)$ converges to 0 on I_n.

Since $\rho(x(t), E_n)$ converges to 0 on I_n, there exists $T > 0$ such that for all $t > T, t \in I_n$, $\rho(x(t), E_n) < L$. But by assumption, for $x \in B(E_n, L) \cap M_n$ the system can not switch from M_n to some M_j, $j \neq i$, which means that the system will stay under the control of g_n for all $t > T$ and therefore converge to E_n. ■

This proposition underlines the difference between our work and other investigations of the stability of hybrid systems (see Section I). In our work, the equilibrium sets of the system on each of the manifolds are in general different, so the condition (1.1) is necessary to guarantee the convergence to the desired equilibrium set. In most other existing literature, the equilibrium point is the same in all the regimes so the trajectory will in general keep crossing different regions as it approaches this point. Schematically, the two alternatives are shown in Figure 4.

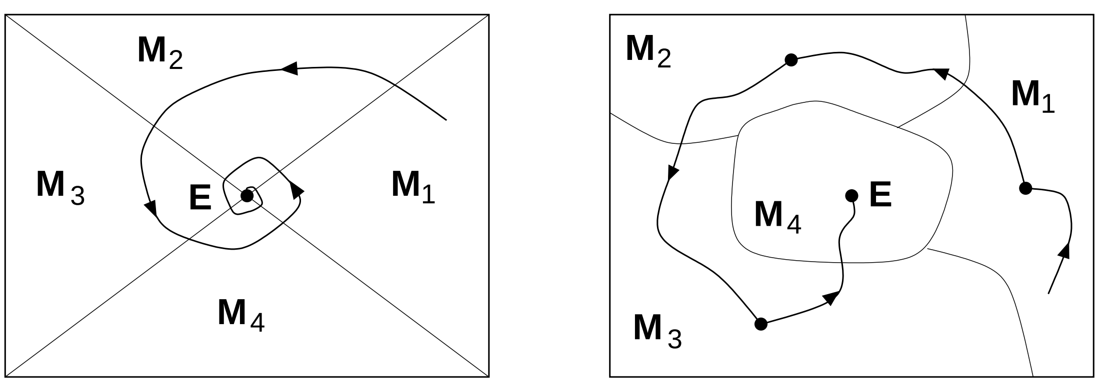

Fig. 4. A schematic of the evolution of a hybrid system with a single equilibrium point (left) and when the equilibrium points are different (right).

While Proposition III.4 gives sufficient conditions for convergence of the system trajectories to E_n, these conditions are difficult to check and therefore not suitable for controller design. It is particularly difficult to check the condition (1.2). By introducing hierarchy among dynamic regimes (continuous controllers), we can obtain conditions that are easier to apply in the design process. The hierarchy will be formally defined through a *partial order*.

Definition III.6: A *(binary) relation* on A is a subset of $A \times A$. If $R(A) \subseteq A \times A$ is a relation, we write aRb or $R(a,b)$ when $(a,b) \in R(A)$. If aRa for all $a \in A$, the relation is *reflexive*. If aRb and $bRc \Rightarrow aRc$, the relation is *transitive*. If aRb and $bRa \Rightarrow a = b$, the relation is *antisymmetric*. If R is a relation, the smallest transitive relation R^{Trans} such that $R \subseteq R^{\text{Trans}}$ is called a *transitive closure* of R.

Definition III.7: A binary relation on a set A that is reflexive, transitive and antisymmetric is called a *partial order* and we will denote it by $\preceq$. If $a \preceq b$ and $a \neq b$, we also write $a \prec b$. If there exists an element $s \in A$ such that $s \preceq x$ for every $x \in A$, s is called the *smallest element*. When it exists, the smallest element is unique.

A relation $R(A)$ can be represented by a (directed) graph with vertices A and edges $R(A)$. For a partial order, transitivity and antisymmetry imply that there will be no nontrivial cycles in the corresponding graph.

Now let $\mathcal{M} = \{M_i\}_{i \in \Xi}$ be the collection of manifolds for a SCD Ω. The switching scheme $\mathcal{S}$ defines a relation $\sigma(\Xi)$, if we put $\sigma(i,j)$ when it is possible to switch from the manifold M_i (controller g_i) to the manifold M_j (controller g_j). More formally:

$$\sigma(\Xi) = \{(i,j) \mid \exists x \in M_i \text{ s.t. } \mathcal{S}(i,x) = j\} \tag{12}$$

Note that the graph representing this relation is precisely the graph described in Section II. We can then show:

Proposition III.8: Take $\sigma^{\text{Trans}}(\Xi)$ and let $\preceq$ be a partial order within $\sigma^{\text{Trans}}(\Xi)$ that has n for the smallest element. Assume the switching scheme $\mathcal{S}$ has the following properties:
(2.1) There exists $L > 0$ such that $\mathcal{S}(n,x) = n$ for every $x \in B(E_n, L) \cap M_n$.
(2.2) If $x(t)$ is a trajectory of Ω and M_i, $i \neq n$ is a manifold on which $x(t)$ evolves for an infinite amount of time, then there exists $\Delta > 0$ such that for every T we can find $\tau > T$ such that $\mathcal{S}(\eta(t), x(t)) \prec i$ for every $t \in [\tau, \tau + \Delta]$.
(2.3) If a system switched from g_n to some other controller at time t_{off} and if t_{on} is the time when the system switches again to g_n, then $V_n(t_{\text{off}}) \geq V_n(t_{\text{on}})$.
Then the submanifold E_n is globally attractive.

Remark III.9: The condition (2.1) is the same as the condition (1.1) in Proposition III.4, while the conditions (2.2) and (2.3) together replace the condition (1.2). The condition (2.2) means that for any manifold M_i on which a trajectory stays for an infinite amount of time, we can find a switch at an arbitrarily large time to a manifold that lies lower in the partial order $\preceq$ and that after such a switch the system evolves on the manifolds that are below M_i for at least Δ.

Proof: We will show that the conditions (2.2) and (2.3) imply the condition (1.2) of Proposition III.4. Let $x(t)$ be a trajectory of Ω and let M_i be a manifold on which $x(t)$ evolves for an infinite amount of time. Since we have a finite number of manifolds, there will be at least one such i. The condition (2.2) guarantees that there will be an infinite number of instances when the system evolves for at least Δ on manifolds that are below

M_i in the hierarchy defined by $\preceq$. But this implies that $x(t)$ will evolve on these manifolds for an infinite amount of time and since there are only finitely many manifolds below M_i, there must exist a manifold M_j with $j \prec i$ on which $x(t)$ evolves for an infinite amount of time. By proceeding recursively and because n is the smallest element for $\preceq$, we conclude that the system must evolve on M_n for an infinite amount of time and in instances that last for at least Δ. The condition (2.3) guarantees that each time the system switches to g_n, the value of the Lyapunov function is smaller than when the system last switched off M_n. The existence of the sequence $\{t_i\}$ in the condition (1.2) of Proposition III.4 is therefore guaranteed. ■

Using Proposition III.8 we can design a stable switching scheme by choosing a partial order, developing controllers on each M_i that guarantee a switch to a lower level with respect to this partial order, and enforcing the decreasing of V_n at switches to M_n. However, developing controllers that guarantee a switch to a lower level is still not an easy task. One possible strategy is to make each controller stabilize a certain manifold within a region from which the system switches to manifolds lower in the hierarchy. A similar idea was employed in [28]. The idea can be formalized in the following way:

Proposition III.10: Assume a partial order $\preceq$ on Ξ that has n for the smallest element. Let each controller g_i asymptotically stabilize a manifold E_i and assume we can find a Lyapunov function V_i for g_i. Let the switching scheme $\mathcal{S}$ satisfy the following conditions:

(3.1) For each i, there exists $L_i > 0$ such that $\mathcal{S}(i, x) \prec i$ for every $x \in B(E_i, L_i) \cap M_i$ (for $i = n$ we require $\mathcal{S}(n, x) = n$).

(3.2) There exists $\Delta > 0$, such that for any T at which the system switches from g_i to g_j, $j \prec i$, $\mathcal{S}(\eta(t), x(t)) \prec i$ for each $t \in [T, T + \Delta]$.

(3.3) If the system switches from g_i to some g_j, $i \prec j$, at time t_{off} and after that switches again to g_i at time t_{on} and if $\mathcal{S}(\eta(t), x(t)) \not\prec i$ for all $t \in [t_{\text{off}}, t_{\text{on}}]$, then $V_i(t_{\text{off}}) \geq V_i(t_{\text{on}})$.

Then the submanifold E_n is globally attractive.

Remark III.11: For $i = n$, the conditions (3.1) and (3.3) above clearly become the same as conditions (2.1) and (2.3) in Proposition III.8. Note that Propositions III.8 and III.10 imply that we can examine the stability of the system by simply examining the relations between the immediate neighbors in the partial order. This has important implications for the synthesis problem and can be exploited to obtain modularity of the controller design process.

Proof: We will show that the above conditions imply the conditions of the Proposition III.8. Assume that a trajectory $x(t)$ evolves on a manifold M_i for an infinite amount of time, but after some time T it never switches to any manifold M_j such that $j \prec i$. Let $I_i = \{t > T \mid \mathcal{S}(\eta(t), x(t)) = i\}$, the union of the intervals beyond T during which the system evolves on M_i. By the condition (3.3), V_i will be monotonically decreasing on I_i and by the condition (3.2), we can find an infinite sequence of (disjoint) intervals of length Δ that lie in I_i. By the same reasoning as in the proof of Proposition III.4, $x(t)$ converges to E_i. By the condition (3.1) this implies that the system will switch to some M_j, $j \prec i$, which is a contradiction. This and the condition (3.2) above therefore imply the condition (2.2) of Proposition III.8. ■

Condition (3.1) is often too stringent, but it is not difficult to see that it can be relaxed:

Corollary III.12: For any i, condition (3.1) in Proposition III.10 can be replaced by the following two conditions:

(3.1a) There exists a submanifold $H_i \subset M_i$ such that any trajectory converging to E_i intersects H_i.
(3.1b) $\mathcal{S}(i, x) \prec i$ for every $x \in H_i$.

So far, we have assumed that the switching scheme $\mathcal{S}$ is static. In other words, $\mathcal{S}$ only depends on the current state (i, x). However, Proposition III.10 suggests that the switching scheme should change so that a switch to some controller g_i should be allowed only at the points where the condition (3.3) is satisfied. For this to be true, the switching scheme in all the discrete states j such that $i \prec j$ should depend on $V_i(t_{\text{off}})$. More generally, we would like to allow the switching scheme to change when the discrete state changes. Such changes can be also understood as adding memory to the discrete controller, so that its actions depend on the past evolution of the system (the discrete state). A look at the proof of Proposition III.8 shows that there is nothing that would require the switching scheme to be static. The proposition therefore remains true even if the switching scheme changes as the discrete state evolves. We have to be slightly more careful with the Proposition III.10. In that case, we need to guarantee that even if the switching scheme changes, the convergence of the trajectories on a manifold M_i to an equilibrium set E_i implies that the system will switch to some g_j, $j \prec i$. We can show:

Proposition III.13: Assume that the switching scheme changes from $\mathcal{S}^-$ to $\mathcal{S}^+$ whenever the system switches from the discrete state i to the discrete state $j \not\preceq i$. If this change is such that the following holds:

$$\mathcal{S}^+(j, x) \neq \mathcal{S}^-(j, x) \;\Rightarrow\; j \not\prec i \text{ and } \mathcal{S}^+(j, x) = j \tag{13}$$

then Proposition III.10 still holds, as long as the condition (3.1) holds for any fixed time.

Note that if we make the switching scheme depend on $V_i(t_{\text{off}})$ as the system leaves g_i in order to satisfy the condition (3.3), the switching scheme will change exactly as above.

Proof: If (13) holds, the switching scheme does not change in any discrete state that is lower (in the partial order) than the current discrete state. Therefore, conditions (3.1) and (3.3) can be employed in the proof of Proposition III.10 in the same way as before. Similarly, the change satisfying (13) does not affect the condition (3.2). ∎

C. Stability

Propositions III.4, III.8, and III.10 provide sufficient conditions for E_n to be attractive, not to be stable. To prove the stability we have to show that trajectories starting outside M_n "nicely" converge to M_n. One possible way of stating this is:

Corollary III.14: The manifold E_n will be stable if in addition to the conditions of Proposition III.4:
(1.3) For any $R > 0$ and every i, there exists $r > 0$ such that if $x(0) \in (M_i \setminus M_n) \cap B(E_n, r)$ then under the control of g_i, $x(t) \in B(E_n, R)$ for every $t > 0$.

Proof: The Lyapunov function V guarantees that for any $R_2 > 0$, there exists $r_2 > 0$ such that $x(0) \in M_n \cap B(E_2, r_2)$ implies $x(t) \in B(E_n, R_2)$ as long as $x(t)$ stays in M_n. Take $R_2 = \min\{R, L\}$ and find the corresponding r_2. Take $R_1 = \min\{R, r_2\}$. By assumption, there exists r_1 such that $x(t)$ stays in $B(E_n, R_1)$ for any trajectory starting in $B(E_n, r_1) \setminus M_n$ and evolving in M_i. By the condition (1.1) of Proposition III.4 and by the choice of R_1, $x(t)$ will intersect M_n inside $B(E_n, r_2) \cap M_n$. But a trajectory on M_n

that comes inside $B(E_n, r_2) \cap M_n$ will stay inside $B(E_n, L) \cap M_n$ and thus remain under the control of g_n (and stay inside $B(E_n, R)$) for all later times. ■

Remark III.15: If we assume the scenario of Proposition III.10 and for every i, $E_i \subseteq E_n$, the condition (1.3) will be trivially true.

IV. Design of switching controllers

Proposition III.10 (Corollary III.12) can be used to design stable switching controllers. Design of the discrete controller will be guided by choosing a partial order on Ξ (hierarchy among M_i's). The partial order can not be chosen arbitrarily, part of it is usually dictated by the physics of the problem (see also Section V). The continuous controllers on each M_i must be then designed so that they are consistent with the chosen partial order. In particular, we need to satisfy the conditions (3.2) and (3.3) of the Proposition III.10. We outline three techniques that can be used to satisfy these conditions.

A. Hysteresis

Hysteresis can be used to enforce the condition (3.2). Suppose we want to switch from M_i to M_j, $j \prec i$. If $f_j(x, g_j(x))$ is bounded for all x in a neighborhood in which the switch occurs, then the condition (3.2) will be automatically satisfied if:

$$\inf_{\{x \mid \mathcal{S}(i,x)=j\}} d(x, \{y \mid \mathcal{S}(j,y) \not\preceq j\}) > d_0$$

where $d_0 > 0$ is some constant. In other words, after the switch to M_j we are some (fixed) finite distance from any point y in H where the system would switch to a discrete state which is not lower in the partial order. Because of the bounded rate of change of the continuous state, this implies that such a switch can only occur after some finite time interval. Hysteresis in the switching rule is also useful to eliminate chattering and increase the robustness of the system.

B. Mixing of the controllers

It is difficult to directly design controllers that would satisfy the condition (3.3). An alternative is to combine several controllers. More precisely, suppose we would like to allow switches from M_i to M_j, $j \prec i$. To satisfy the condition (3.3), we need to have a controller g_i that is able to decrease the Lyapunov function V_j. Controllers g_i and g_j stabilize E_i and E_j, respectively. We also know that the controller g_j decreases the Lyapunov function V_j. If $E_j \subset E_i$, we can construct a new controller, $\hat{g}_i$ that brings the system to E_i and then moves the state along E_i towards E_j. Chose some neighborhood H of E_i and define $\hat{g}_i$ to be:

$$\hat{g}_i(x) = \begin{cases} g_i(x) & \text{if } x \in M_i \setminus H \\ g_i(x) + \epsilon g_j(x) & \text{if } x \in M_i \cap H \end{cases} \tag{14}$$

where ϵ is a small constant. Now we can modify the switching scheme as discussed in Proposition III.13 so that we do not switch from g_i to g_j until V_j has sufficiently decreased. This will guarantee that the condition (3.3) is satisfied. The following lemma gives rigorous justification for such a design:

Lemma IV.1: Let E be an asymptotically stable manifold for the system:

$$\dot{x} = f(x) \tag{15}$$

and let V be a Lyapunov function satisfying Theorem III.3 such that $|\frac{\partial V}{\partial x}|$ is bounded on $B(E,R)$ for some R. Let $g(x)$ be a function such that $|g(x)|$ is bounded on $B(E,R)$. Given $0 < L < R$, it is always possible to choose $\epsilon > 0$ such that every trajectory of the system:

$$\dot{x} = f(x) + \epsilon\, g(x) \tag{16}$$

that starts in $B(E,L)$ never leaves it.

Proof: Let $\dot{V}_u$ be the derivative of V along the trajectories of 15 and $\dot{V}_p$ the derivative of V along the trajectories of 16. Let $Q = \sup_{B(E,L)} |\frac{\partial V}{\partial x}|$ and $M = \sup_{B(E,L)} |g(x)|$. Choose $\epsilon = \frac{\gamma^{-1}(L)}{QM}$ (where γ is a monotonically increasing function such that $\dot{V}_u(x) \leq -\gamma(\rho(x,E))$, see Theorem III.3). Then we have:

$$\dot{V}_p(x(t)) = \dot{V}_u(x(t)) + \frac{\partial V}{\partial x}\epsilon g(x) \leq -\gamma(\rho(x,E)) + \epsilon QM < 0 \;\text{ if }\; \rho(x,E) > L \tag{17}$$

■

The Lemma tells us that it is always possible to construct the controller $\hat{g}_i(x)$ so that once the system enters the ball $B(E_i, L_i)$, where L_i is the constant from the condition (3.1), it will not leave the ball until it switches to g_j. To verify that $\hat{g}_i$ actually decreases V_j while the system evolves on $B(E_i, L_i)$, we can use linearization around E_i.

C. Dealing with autonomous switches

Mixing of the controllers can be used to enforce the condition (3.3) when the switches between two discrete states i and j are under our control. But there will be situations when the system switches autonomously. In this case, it is helpful to decouple the autonomous switches from the switches between controllers with different equilibrium sets. Assume the system can switch autonomously from the discrete state i to j, but the controllers g_i and g_j have different equilibrium sets. We can try to design a controller g_j^i on M_j that has the same equilibrium set and the same Lyapunov function as the controller g_i on M_i. Then instead of switching from g_i to g_j, we switch from g_i to g_j^i. Since the controllers have the same Lyapunov function, the condition (3.3) is automatically satisfied. To switch to the controller g_j (and enforce the convergence to a different equilibrium set), we then use mixing of the controllers g_j^i and g_j, as described above.

V. Example: Hilare robot

We study control of a mobile robot of a Hilare type. A schematic of such a robot is shown on Fig. 5. The robot has two independently actuated wheels. The inputs to the system are the two torques that drive the wheels. In the robotics literature, it is commonly assumed that the wheels of the robot do not slip, leading to a nonholonomic system. However, slippage of the wheels can easily occur if the robot drives on a slippery surface, when large torques are applied on the wheels, when the robot makes turns at sufficiently high velocity, or when it is perturbed by external forces. As the experience with cars shows, slippage of the wheels can lead to catastrophic results. In this example, we therefore study control of a Hilare robot whose wheels can slip. Since the dynamics of the robot change as the wheels switch between rolling and sliding, this is an example of a SCD. We want to make the robot drive along a prescribed line in the plane with a constant forward velocity $v_0 > 0$ (such a line is a relative equilibrium). Referring to Fig. 5, the control task will be to stabilize the robot to the line $y = 0$ and the velocity $v_x = v_0$.

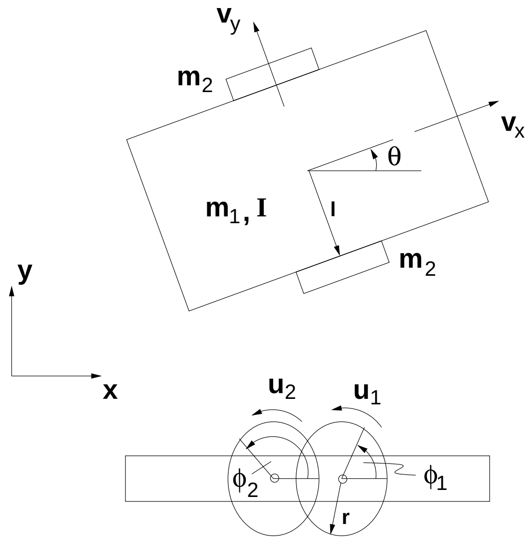

Fig. 5. A top view and a side view of the Hilare robot

The dynamics of the system changes depending on whether the wheels are rolling or sliding. We have four different regimes: (a) both wheels are rolling; (b) both wheels are sliding; (c) wheel 1 is rolling and wheel 2 is sliding; (d) wheel 1 is sliding and wheel 2 is rolling. We denote these regimes by RR, SS, RS and SR, respectively. To each regime corresponds a manifold on which the system evolves. The graph describing the system's discrete behavior is shown in Fig. 6.

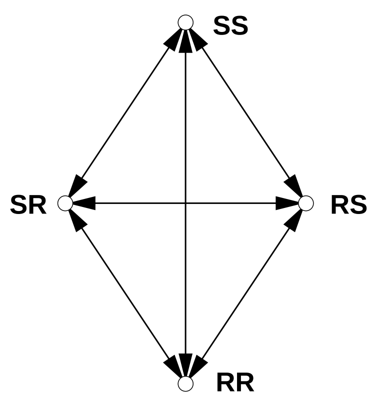

Fig. 6. Graph corresponding to a Hilare robot

In general, the Hilare robot can be described with 5 configuration variables: position of the center of mass (x and y) and orientation of the main body (θ), and the angles of rotation of both wheels (ϕ_1 and ϕ_2). The phase space is therefore in general of dimension 10. If any of the wheels is rolling, additional relations among the state variables must hold and the dimension of the state space decreases.

In order to derive the dynamic equations, we first write down the expression for the kinetic energy of the system. It is useful to express the kinetic energy in the body-fixed frame:

$$2\,T = w_1^2 I_w + w_2^2 I_w + \omega^2 (I_b + 2I_v) + (v_x^2 + v_y^2)(m_b + 2m_w) \tag{18}$$

The symbols are:

I_w	Moment of inertia of a wheel around its axis
I_b	Moment of inertia of the body of the robot around the vertical axis through the center of mass
I_v	Moment of inertia of a wheel around the vertical axis through the center of mass
m_b	mass of the body of the robot
m_w	mass of a wheel

while the variables are:

w_1	rotational velocity of the wheel 1
w_2	rotational velocity of the wheel 2
ω	rotational velocity of the body of the robot
v_x	forward velocity
v_y	lateral velocity

It is important to observe that the expression for the kinetic energy does not depend on the position of the robot in space or on the angles of rotation of the wheels. The same is true for the actuating torques, as well as the forces between the ground and the wheels. In the language of mechanics, we say that the system is invariant under the action of a group $SE(2) \times S^1 \times S^1$. Since this group is also the configuration manifold for the system, we can use the Euler-Poincaré equations to describe the dynamics of the system [33]. The equations are:

$$\begin{aligned}
\dot{w}_1 &= \frac{F_x^1 r + u_1}{I_w} \\
\dot{w}_2 &= \frac{F_x^2 r + u_2}{I_w} \\
\dot{\omega} &= l \frac{F_x^1 - F_x^2}{I_b + 2I_v} \\
\dot{v}_x &= \omega\, v_y + \frac{F_x^1 + F_x^2}{m_b + 2m_w} \\
\dot{v}_y &= -\omega\, v_x + \frac{F_y^1 + F_y^2}{m_b + 2m_w}
\end{aligned} \tag{19}$$

The configuration (group) variables can be obtained from the velocities by integrating the equation:

$$\dot{g} = g\xi \tag{20}$$

where $g \in SE(2) \times S^1 \times S^1$ stands for the group variables and ξ for the body-velocity (Lie algebra) variables [33]. In this paper, we are only interested in controlling the configuration variables y and θ, the rest of the variables are controlled at the velocity level. The dimension of the system is thus 7. The forces F^1 and F^2 are the forces between the ground and the two wheels, expressed in the body-fixed frame. The indexes x and y stand for the forward and lateral directions. The rest of the symbols are explained in Fig. 5.

If the wheel 1 is rolling, we have two constraints:

$$\begin{aligned} v_x + l\omega + rw_1 &= 0 \\ v_y &= 0 \end{aligned} \tag{21}$$

Similarly, when the wheel 2 is rolling, the constraints are:

$$\begin{aligned} v_x - l\omega + rw_2 &= 0 \\ v_y &= 0 \end{aligned} \tag{22}$$

When the wheel i is rolling, the force F^i prevents slippage of the wheel and can be eliminated from Eq. (20) using Eq. (21) or Eq. (22). Since the rolling constraint results in two constraint equations, the dimension of the system drops by 2 whenever a wheel starts rolling. However, when both wheels are rolling, (21) and (22) only constitute 3 independent constraints. Therefore, when both wheels are rolling, the dimension of the system is 4, including the configuration variables y and θ.

When the wheels are sliding, the reaction forces are frictional forces. The force F^i when the wheel i is sliding is:

$$F^i = -\mu_d \frac{v_r^i}{\|v_r^i\|}(\frac{m_b}{2} + m_w)g \tag{23}$$

where μ_d is the coefficient of (dynamic) friction, g is the gravity constant and v_r is the relative velocity between the point on the wheel which is in contact with the ground and the ground (the left-hand sides of Eqs. (21-22)).

The switch from rolling to sliding occurs when the amplitude of the constraint force exceeds the amplitude of the (static) friction:

$$\|F^i\| > \mu_s(\frac{m_b}{2} + m_w)g \quad \Rightarrow \quad \text{rolling} \rightarrow \text{sliding} \tag{24}$$

The condition for the switch from sliding to rolling is that the relative velocity is 0 and that the amplitude of the frictional force is greater than the amplitude of the constraint force:

$$v_r^i = 0 \ \text{ and } \ \|F^i\| \leq \mu_s(\frac{m_b}{2} + m_w)g \quad \Rightarrow \quad \text{sliding} \rightarrow \text{rolling} \tag{25}$$

A. Controller design

The control task is to stabilize the robot to the relative equilibrium $E_{RR} = \{y = 0,\ v_x = v_0\}$, a subset in the regime RR. The control strategy will be to first make the wheels roll and then stabilize the robot to the desired relative equilibrium. Note that once the wheels roll, we can not guarantee that no slippage will occur, the convergence to the desired set must be attained through switching.

When the wheels roll, the constraint force also depends on the control inputs. Therefore, we do have a certain amount of control over the switches from rolling to sliding. However, once the feedback controllers for u_1 and u_2 are designed, the switches between rolling and sliding are autonomous (the discrete inputs have no effect on the switches). To deal with these autonomous switches we can use the method described in Section IV-C. We need to design a set of controllers for the four regimes that share a Lyapunov function at least pairwise. This is possible since the equations for w_1 and w_2 can be feedback linearized regardless whether F is a constraint force or a friction force. Let the linearizing controllers obtained by substituting the appropriate expressions for the force F in different regimes be g_{SS}, g_{SR}, g_{RS}, and g^1_{RR} (the last superscript indicates that we will need additional controllers in the regime RR). Using these controllers, we can drive the wheels to a constant value $-\frac{v_0}{r}$ (the nominal driving velocity). Given that friction is a dissipative force, we expect that after w_1 and w_2 converge, the wheels of the robot will start rolling, causing the robot to drive in a straight line. The formal proof that the controllers g_{SS}, g_{SR}, g_{RS}, and g^1_{RR} drive the system to the regime RR and asymptotically stabilize the set $E^1_{RR} = \{w_1 = w_2 = -\frac{v_0}{r},\ v_x = v_0,\ v_y = \omega = 0\}$ is deferred to Appendix A.

The next step is to design a controller in the regime RR that stabilizes the robot to the desired equilibrium set. We choose v_x and ω as the velocity variables (the others are eliminated through the nonholonomic constraints), so the state of the system in the regime RR is (y, θ, v_x, ω). Consider the following two outputs:

$$h_1 = y + L\sin\theta \tag{26}$$

$$h_2 = v_x \tag{27}$$

where L is an arbitrary positive constant. Physically, h_1 corresponds to the y coordinate of a point displaced by L along the x axis of the body-fixed reference frame, and h_2 to the forward velocity of the vehicle. Using these two outputs, we can design a controller that input-output linearizes the system (if $L \neq 0$). We can therefore make the set $\dot{x} = 0$, $y = -L\sin\theta$ globally asymptotically stable. Such controller was proposed in [34] for trajectory tracking. Let this controller be g^3_{RR}. With the chosen outputs, the relative degree of the system is $(2, 1)$, implying that the system has one-dimensional (nonlinear) internal dynamics. The state corresponding to the internal dynamics can be chosen to be y, so the internal dynamics becomes:

$$\dot{y} = h_2 \sin\theta = h_2 \frac{h_1 - y}{L} \tag{28}$$

The equilibrium sets for the system are therefore $h_1 = \dot{h}_1 = y = 0$, $h_2 = v_0$. Note that this equilibrium set contains values $\theta = k\,\pi$ for any $k \in \mathbb{Z}$. To analyze the stability properties of the system, consider the function:

$$V_3 = V_3^h + y^2$$

where V_3^h is the Lyapunov function corresponding to the linearized dynamics such that $\dot{V}_3^h = -(h_2 - v_0)^2 - \frac{2\,v_0}{L} h_1^2 - \dot{h}_1^2$. A closer look at $\dot{V}_3$ reveals that it vanishes on the equilibrium set of the system and is negative for any trajectory starting with $h_2 > 0$. Furthermore, the level sets $V_3 = C$ are closed around E_{RR} (the value $\theta = 0$) for C small enough. According to LaSalle's theorem, for any such C, the set $W = \{(y, \theta, v_x, \omega) \,|\, V_3(y, \theta, v_x, \omega) < C\}$ is therefore the region of attraction of E_{RR}.

To guarantee that the system converges to $\theta = 0$ it is thus necessary to bring the trajectory of the system into W. On the other hand, the controllers g_{SS}, g_{SR}, g_{RS}, and g_{RR}^1 stabilize the robot to the set $E_{RR}^1 = \{v_x = v_0, v_y = \omega = 0\}$, so the value of θ and y can be arbitrary. For that reason, we need to design an additional controller, g_{RR}^2. We proceed similarly as before. We select the outputs we wish to control:

$$h_1 = \theta \tag{29}$$

$$h_2 = v_x \tag{30}$$

and obtain the controller g_{RR}^2 by input-output linearization. The Lyapunov function V_2 for the linearized system is easy to construct. The zero dynamics of the system is $\dot{y} = 0$.

In total, we have designed six controllers: controllers g_{SS}, g_{SR} and g_{RS} in regimes SS, SR and RS, respectively, and three controllers, g_{RR}^1, g_{RR}^2 and g_{RR}^3 in the regime RR. In order to use the model described in Section II, we replace the manifold M_{RR} with the manifolds M_{RR}^1, M_{RR}^2 and M_{RR}^3 (all equal to M_{RR}). The final partial order between the controllers is shown in Fig. 7.a: the partial order between the controllers g_{SS}, g_{SR}, g_{RS} and g_{RR}^1 is induced by the physics of the problem, while the rest of the partial order follows from the way we designed g_{RR}^1, g_{RR}^2 and g_{RR}^3.

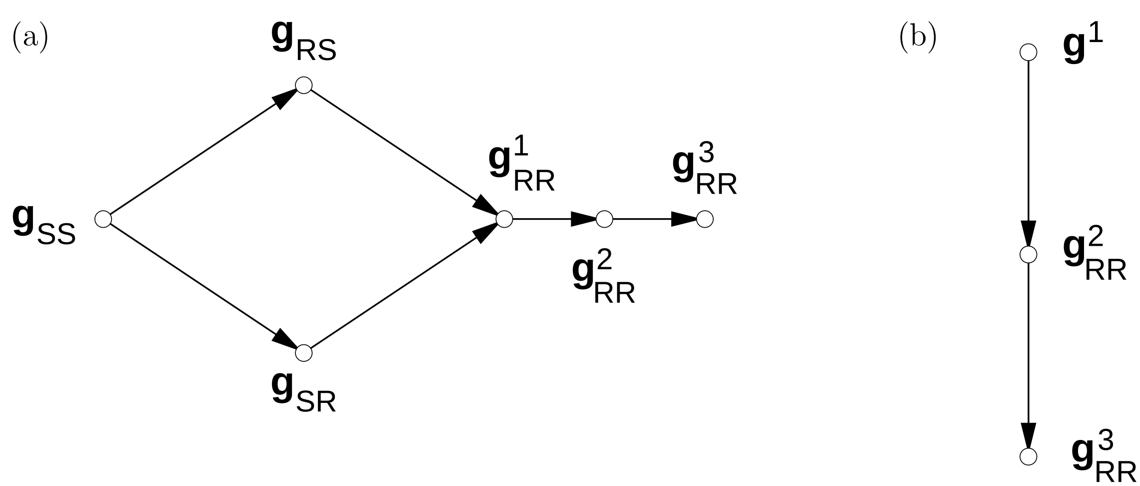

Fig. 7. (a) Partial order between all the controllers; (b) partial order after abstraction of controllers g_{SS}, g_{SR}, g_{RS} and g_{RR}^1 into g^1.

B. Convergence to E_{RR}

To show the convergence of the robot to the desired equilibrium set E_{RR}, we will use Proposition III.10 (and Corollary III.12). We could proceed by simply checking the conditions for all six controllers. However, given that the controllers g_{SS}, g_{SR}, g_{RS}, and g_{RR}^1 all stabilize E_{RR}^1 and make the wheels rolling, we can avoid analyzing these four controllers again. Instead, we consider them as a single, more complicated, controller g^1 that stabilizes $E^1_{RR} \subset M^1_{RR}$. This shows that the partial order concept is very convenient for designing switching controllers since it naturally leads to a modular design.

Consider therefore the controllers g^1, g^2_{RR} and g^3_{RR}. The resulting partial order is shown in Fig. 7.b. We must also design a switching scheme (discrete controller). Consider the following function:

$$\mathcal{S}(\eta, x) = \begin{cases} 3 & \eta = 3 \text{ and } |\theta| < \frac{\pi}{2} \text{ and } \|F_c^1\| < F_g \text{ and } \|F_c^2\| < F_g \quad (1) \\ 3 & |\theta| < \lambda\frac{\pi}{2} \text{ and } |v_x| < v_0 \quad (2) \\ & \quad \text{and } \|F_c^1\| < \lambda F_g \text{ and } \|F_c^2\| < \lambda F_g \\ 2 & \eta = 2 \text{ and } \|F_c^1\| < F_g \text{ and } \|F_c^2\| < F_g \quad (3) \\ 2 & \|F_c^1\| < \lambda F_g \text{ and } \|F_c^2\| < \lambda F_g \quad (4) \\ 1 & \eta = 1 \text{ and } \|F_c^1\| < F_g \text{ and } \|F_c^2\| < F_g \quad (5) \end{cases}$$

where $0 < \lambda < 1$ and $F_g = \mu_d(\frac{m_b}{2} + m_w)g$. The above conditions have to be evaluated in order: first we select the controller g^3_{RR} and compute the corresponding constraint forces (the constraint forces depend on the input torques and thus the controller). If neither of the conditions (1) and (2) is satisfied, we try with the controller g^2_{RR} and test conditions (3) and (4). If also these conditions fail, we use the controller g^1 (at this step the system might switch to one of the regimes SS, SR or RS).

The factor λ in conditions (2) and (4) achieves hysteresis in the switches. Condition (2) implies that when the system switches to g^3_{RR} from g^2_{RR}, the constraint force will be strictly less than the maximum allowed value F_g (condition (3)). Since the rate of change of the constraint force is bounded, the system is guaranteed to stay under the control of g^3_{RR} for some finite amount of time. Condition (4) similarly implements hysteresis in switches from g^1 to g^2_{RR}. The requirement $\theta < \frac{\pi}{2}$ in conditions (1) and (2) guarantees that the controller g^3_{RR} stabilizes the system to the value $\theta = 0$ (as opposed to $\theta = \pi$). Note that the factor λ is used in condition (2) to achieve hysteresis. Finally, the requirement $|v_x| < 2\,v_0$ in condition (2) guarantees that $\dot{V}_3$ is negative when the system is controlled by g^3_{RR}.

We next show that all the conditions of Proposition III.10 are satisfied. Since the dynamics do not change as the system switches between the controllers g^1, g^2_{RR}, and g^3_{RR}, we use mixing of controllers as described in Section IV-B to satisfy the condition (3.3). The condition (3.2) is satisfied by employing hysteresis in $\mathcal{S}$ through the factor λ (Section IV-A). Finally, by appropriately choosing the gains of the linearizing controllers g^2_{RR} and g^3_{RR}, the constraint forces on the equilibrium manifolds E^1_{RR} and E^2_{RR} can be made strictly less than λF_g. The condition (3.1) (conditions (3.1a) and (3.1b) for controller g^1) is thus satisfied in all three discrete states.

C. Simulation results

Figure 8 shows snapshots of a simulation run. The figures are shown at time intervals equal to $0.44s$. The whole sequence spans $8s$. The numbers in the figures represent the value of the switching function (controllers that were active): $3, 2, 1, -1, -2, -3$ correspond to g^3_{RR}, g^2_{RR} and g^1_{RR}, g_{RS}, g_{SR}, and g_{SS}, respectively. Figure 9 shows the trajectories for y, θ and $\dot{x}$ together with the value of the switching function. The initial velocities of the wheels (w_1 and w_2) were set to 0 while the velocities of the body of the vehicle were set to

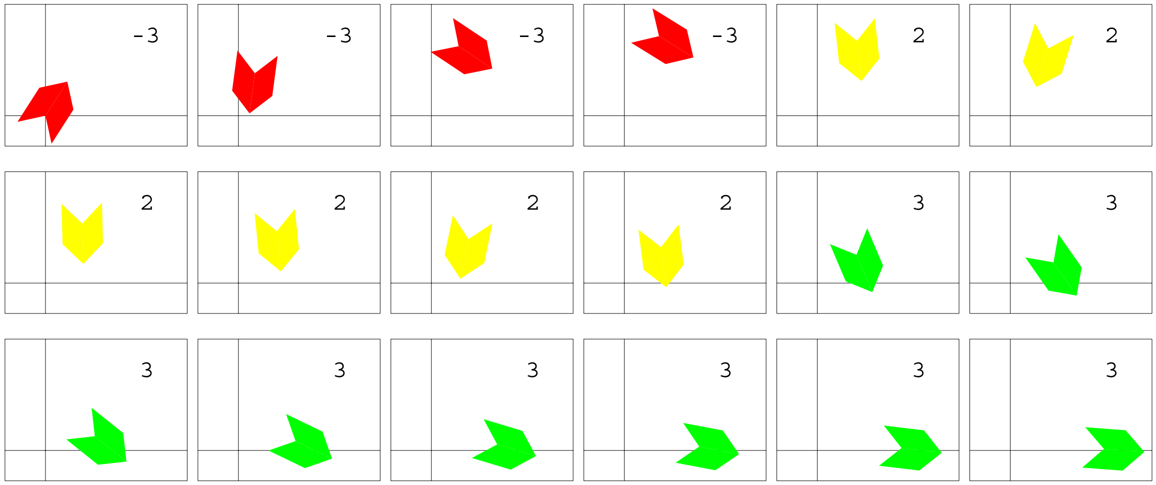

Fig. 8. Snapshots of the motion of the robot.

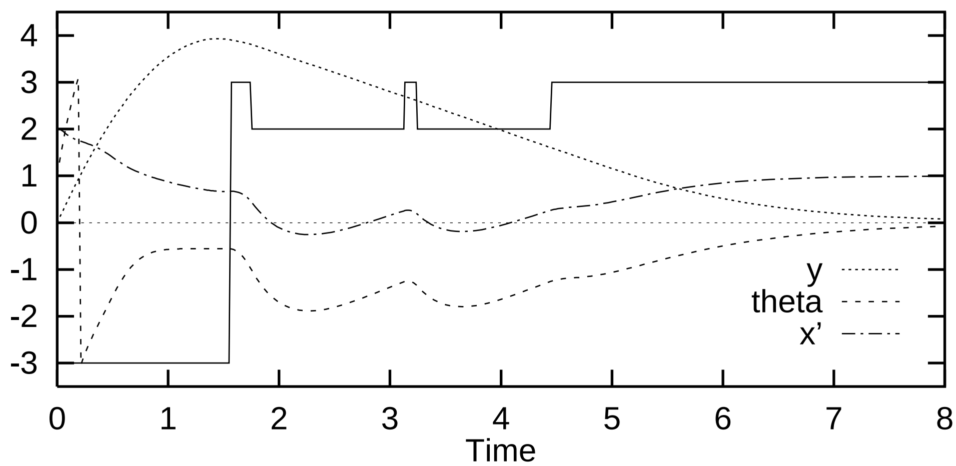

Fig. 9. Trajectories for y, θ, and $\dot{x}$. The solid line is the value of switching function.

$(\dot{x}, \dot{y}, \dot{\theta}) = (2, 2, 15)$. These initial velocities correspond to large initial relative velocities, giving rise initially to a period of sliding and a large excursion in the y direction. During the sliding phase the robot also makes a full turn around its center. Both wheels stop sliding at time $1.57s$ and the system switches to the controller g^3_{RR}. Two switches between g^3_{RR} and g^2_{RR} follow: at $1.75s$ and $3.25s$. The corresponding switches from g^2_{RR} to g^3_{RR} occur at $3.13s$ and $4.46s$. From time $4.46s$ on, the controller g^3_{RR} remains active and it finally stabilizes the robot to E_{RR}.

VI. Conclusion

We investigated the problem of stabilizing systems with changing dynamics (SCD's). Such systems form a subclass of hybrid systems. A natural control strategy for stabilization of a SCD is to design a switching controller consisting of continuous controllers for each dynamic regime and a discrete controller that switches between them. We derived sufficient conditions for stability of a switching controller. Using the concept of partial order to introduce the hierarchy among continuous controllers, we have shown that the sufficient conditions can be reformulated in a way that naturally leads to a design methodology. In this framework, the partial order plays the role of the control strategy at the discrete level. We described three strategies that further simplify design of switching controllers: hysteresis in the switching rules, mixing of controllers in the same dynamic regime and decoupling of autonomous switches from the switches in the control objective. These techniques were applied to stabilization of a mobile robot of Hilare type whose wheels are allowed to slip. In the example, we demonstrated that the proposed methodology naturally leads to a modularity in the design process.

The presented work leaves some important questions unanswered. The proposed framework can only be used to control the system towards a time invariant submanifold of the state space, while in many problems it is necessary to stabilize the system to a periodic orbit that passes through several discrete states (example is walking). Another interesting question is whether we could formulate an inverse theorem and show that the partial order is an inherent feature of every stable switching controller. Finally, we showed that mixing of controllers naturally leads to time-varying discrete controllers where memory is used to determine the current switching rules. An interesting problem is what classes of SCD's can be stabilized by purely state-dependent (memoryless) controllers and in which cases memory variables are essential for stabilization.

Appendix

I. Convergence to the rolling regime

A. Partial feedback linearization

A useful technique when designing controllers for nonlinear systems is partial feedback linearization. The following result will prove useful in the ensuing stability analysis:

Lemma A.1: Let

$$\dot{\xi} = \varphi(\xi)$$

be asymptotically stable to $\xi = 0$. Let

$$\dot{x} = f(x) \tag{31}$$

be asymptotically stable to E and $V(x)$ be a Lyapunov function that satisfies Theorem III.3. Let $\Phi(x, \xi)$ be a C^1 function such that $\Phi(x, 0) = 0$. If for any initial condition (x_0, ξ_0) the trajectories of the system:

$$\dot{x} = f(x) + \Phi(x, \xi) \tag{32}$$

remain bounded in x, the system (32) is asymptotically stable.

Proof: Let $\dot{V}_u$ be the derivative of V along the trajectories of (31) and $\dot{V}_p$ the derivative of V along the trajectories of (32). Since the dynamics for ξ is stable and x remains bounded, there exists a ball $B(0, R)$ (in the combined state space (ξ, x)) such that the trajectories $(\xi(t), x(t))$ always stay within. Let $Q = \sup_{B(0,R)} |\frac{\partial V}{\partial x}|$ and $M = \sup_{B(0,R)} \frac{\partial \Phi}{\partial \xi}$. Since V satisfies Theorem III.3, there exist monotonically increasing functions β and γ such that $V(x) \leq \beta(\rho(x, E))$ and $\dot{V}_u(x) \leq -\gamma(\rho(x, E))$. Then we have:

$$\begin{aligned} \dot{V}_p(x(t)) &= \dot{V}_u(x(t)) + \frac{\partial V}{\partial x}\Phi(x, \xi) \\ &\leq -\gamma(\rho(x, E)) + QM\|\xi\| \leq -\gamma(\beta^{-1}(V_p(x))) + QM\|\xi\| \end{aligned} \tag{33}$$

Using the comparison principle [35], we conclude that $V \to 0$. ■

B. Equilibrium sets for g_{SS}, g_{SR}, g_{RS}, and g^1_{RR}

To analyze the convergence properties of g_{SS}, g_{SR}, g_{RS}, and g^1_{RR}, we need to identify the equilibrium sets (sets where the vector fields describing the dynamics vanish) for each of the regimes. We start with the regime g_{SS}. Since we can specify the dynamics for w_1 and w_2, we can make $-\frac{v_0}{r}$ to be the (globally asymptotically stable) equilibrium point for these two states. We next examine the zero dynamics of the system (the remaining dynamic equations, where we assume $w_1 = w_2 = -\frac{v_0}{r}$). We substitute Eqs. (21) and (22) into Eq.

(23) to obtain the frictional force. The zero dynamics of the robot is thus given by:

$$\begin{aligned}
\dot{v}_x &= \omega\, v_y + C_T \left(-\frac{(v_x - v_0) + l\omega}{\sqrt{v_y^2 + (v_x - v_0 + l\omega)^2}} - \frac{(v_x - v_0) - l\omega}{\sqrt{v_y^2 + (v_x - v_0 - l\omega)^2}} \right) \\
\dot{v}_y &= -\omega\, v_x + C_T \left(-\frac{v_y}{\sqrt{v_y^2 + (v_x - l\omega - v_0)^2}} - \frac{v_y}{\sqrt{v_y^2 + (v_x + l\omega - v_0)^2}} \right) \\
\dot{\omega} &= C_R \left(-\frac{l\omega + (v_x - v_0)}{\sqrt{v_y^2 + (l\omega + v_x - v_0)^2}} - \frac{l\omega - (v_x - v_0)}{\sqrt{v_y^2 + (l\omega - v_x + v_0)^2}} \right) \qquad (34)
\end{aligned}$$

where

$$\begin{aligned}
C_T &= \frac{g\mu_d}{2} \\
C_R &= \frac{gl(m_b + 2m_w)\mu_d}{2(I_b + 2I_v)}
\end{aligned}$$

It is not difficult to see that $v_x = v_0$, $v_y = \omega = 0$ is an equilibrium point (fixed point) for the zero dynamics. In the rest of the section we show that this point is also globally asymptotically stable.

We introduce the following symbols:

$$\begin{aligned}
\nu_x &= v_x - v_0 \\
T_1 &= \frac{l\omega + \nu_x}{\sqrt{v_y^2 + (l\omega + \nu_x)^2}} \\
T_2 &= \frac{l\omega - \nu_x}{\sqrt{v_y^2 + (l\omega - \nu_x)^2}}
\end{aligned}$$

Now consider the function $V_\omega = \frac{1}{2}\omega^2$. From Eq. (34) we get:

$$\dot{V}_\omega = -C_R\, \omega\, (T_1 + T_2) \qquad (35)$$

Note that either ν_x or $-\nu_x$ agrees in sign with ω. Assume for example that $\omega\nu_x > 0$. A short calculation then shows that $|T_1| \geq |T_2|$, which implies that $\dot{V} \leq 0$. We therefore conclude that the amplitude of ω decreases with time.

Next, we show that for any initial conditions (even when w_1 and w_2 are not equal to $-\frac{v_0}{r}$) the system remains in a bounded region. Consider the following function:

$$V_{v_x v_y \omega} = \frac{1}{2C_T}(v_x^2 + v_y^2) + \frac{1}{2C_R}\omega^2$$

The derivative of $V_{v_x v_y \omega}$ along the trajectories of (20) is:

$$\begin{aligned}\frac{d}{dt}V_{v_x v_y \omega} &= -\sqrt{v_y^2+(v_x-w_1+l\omega)^2}-\sqrt{v_y^2+(v_x-w_2-l\omega)^2} \\ &\quad +w_1\frac{v_x-w_1+l\omega}{\sqrt{v_y^2+(v_x-w_1+l\omega)^2}}+w_2\frac{v_x-w_2-l\omega}{\sqrt{v_y^2+(v_x-w_2-l\omega)^2}}\end{aligned}$$

Since w_1 and w_2 are directly controlled, we can assume that they are bounded, $|w_i| < W$. We have:

$$\frac{d}{dt}V_{v_x v_y \omega} \leq -\sqrt{v_y^2+(v_x-w_1+l\omega)^2}-\sqrt{v_y^2+(v_x-w_2-l\omega)^2}+2W$$

It is now not difficult to see that $\frac{d}{dt}V_{v_x v_y \omega}$ remains negative outside a ball $B_{v_x v_y \omega}(0, 2\sqrt{3}\,W)$, which implies that v_x, v_y and ω remain bounded.

We can also establish a lower bound on $\dot{V}$ as a function of ω and v_y. First, consider the case $|\nu_x| \leq |l\omega|$. In this case T_1 and T_2 are of the same sign. Therefore:

$$\begin{aligned}|T_1+T_2| &= |T_1|+|T_2| \geq \frac{|l\omega+\nu_x|}{M}+\frac{|l\omega-\nu_x|}{M} = \\ &= \frac{2l|\omega|}{M}\end{aligned}$$

where $M = \max_{\bar{B}(0,2\sqrt{3}\,W)}\{\sqrt{v_y^2+(l\omega+\nu_x)^2}, \sqrt{v_y^2+(l\omega-\nu_x)^2}\}$ (both functions are continuous and therefore bounded on a closed set). Now consider the case $|\nu_x| > |l\omega|$. Define:

$$\Phi(x) = \frac{x+\nu_x}{\sqrt{v_y^2+(x+\nu_x)^2}}$$

where we assume $v_y \neq 0$ is a constant. We have:

$$\Phi'(x) = \frac{v_y^2}{\sqrt{v_y^2+(x+\nu_x)^2}}$$

The mean value theorem then leads to:

$$|T_1+T_2| = |\Phi(l\omega)-\Phi(-l\omega)| \geq 2l|\omega|\frac{v_y^2}{M}$$

In summary, $|\dot{V}_\omega| \leq -l\frac{|\omega|^2}{M}\min\{1, v_y^2\}$.

Now consider the function V_ω again. Since it is non-increasing and bounded from below, it converges to some value $V_\omega^0 = \frac{1}{2}{\omega^0}^2$. First, we claim that this implies that v_y converges to 0. If not, we can find a constant $1 > \epsilon > 0$ and an infinite sequence of times $\{t_i\}$ (where $t_i < t_j$ for $i < j$) such that $|v_y| > 2\epsilon$. Since the rate of change of v_y is limited

($\dot{v}_y \leq |\omega_0|(|v_0| + l|\omega_0|) + 2C_T$), there exists $\eta > 0$ such that $|v_y| > \epsilon$ on every interval $[t_i - \eta, t_i + \eta]$. This implies that on every interval $[t_i - \eta, t_i + \eta]$, $|\dot{V}_\omega| \geq 2l \frac{V\omega^0}{M} \epsilon^2$. It follows:

$$
\begin{aligned}
|V_\omega(t_i + \eta) - V_\omega(t_i - \eta)| &= |\int_{t_i-\eta}^{t_i+\eta} \dot{V}_\omega(t)dt| = \int_{t_i-\eta}^{t_i+\eta} |\dot{V}_\omega(t)|dt \geq \\
&\geq \int_{t_i-\eta}^{t_i+\eta} (2l\frac{V_\omega^0}{M}\epsilon^2)dt \geq 4l\frac{V_\omega^0}{M}\epsilon^2\eta
\end{aligned}
$$

which clearly contradicts the fact that $V_\omega(t)$ converges.

Assuming that v_y is arbitrary small, similar reasoning as for V_ω shows that the function $V_{\nu_x} = \frac{1}{2}\nu_x^2$ is decreasing and it thus converges to a value $V_{\nu_x}^0 = \frac{1}{2}(\nu_x^0)^2$. Equations (34) thus become:

$$
\begin{aligned}
\dot{v}_x &\approx C_T\left(-\text{sgn}(\nu_x^0 + l\omega^0) - \text{sgn}(\nu_x^0 - l\omega^0)\right) \\
\dot{\omega} &\approx C_R\left(-\text{sgn}(\nu_x^0 + l\omega^0) + \text{sgn}(\nu_x^0 - l\omega^0)\right)
\end{aligned}
$$

The expressions on the right are constant and since both ω and ν_x converge, they must be 0. This implies $\nu_x^0 = \omega^0 = 0$.

This shows that $v_y = \nu_x = \omega = 0$ is (globally) asymptotically stable equilibrium set for the zero dynamics. The fact that the zero dynamics is globally asymptotically stable does not necessarily imply that the whole system is. However, since we have shown that any trajectory (v_x, v_y, ω) remains in a bounded region, we can employ Lemma A.1 to conclude that the equilibrium set $E_{RR}^1 = \{w_1 = w_2 = -\frac{v_0}{r},\ v_y = \nu_x = \omega = 0\}$ is globally asymptotically stable.

We can also prove that E_{RR}^1 is globally asymptotically stable equilibrium set in the regimes SR, RS, and RR. But unlike the proof above, the proof for these regimes is simply an algebraic exercise since the dynamic equations become much simpler.

C. Convergence

We have shown that the controllers g_{SS}, g_{SR}, g_{RS}, and g_{RR}^1 all have the same equilibrium set E_{RR}^1. To show that this also implies that the system converges to the dynamic regime RR, we employ Proposition III.8. We want to show that the system converges to the submanifold $E_{RR}^1 \subset M_{RR}$. The partial order among the controllers is shown in Fig. 10. The switching rule is given by the physics of rolling and sliding (Eqs. (24) and (25)). We first need to find a Lyapunov function V for g_{RR}^1 on M_{RR}. This is straight forward, since the dynamics on M_{RR} is given by the dynamics of w_1 and w_2 (because of the constraints (21) and (22)), which are directly controlled:

$$
V = (w_1 - v_0)^2 + (w_2 - v_0)^2
$$

By construction, w_1 and w_2 are also the controlled outputs in all the other regimes. It follows that V will be a (partial) Lyapunov function on the whole space, implying that the condition (2.3) will be automatically satisfied.

Next, we observe that on E_{RR}^1, the constraint force is 0. Therefore, for any $\epsilon > 0$, there is a neighborhood around E_{RR}^1 in which $\|F\| < \epsilon$. This implies that the condition (2.1) holds.

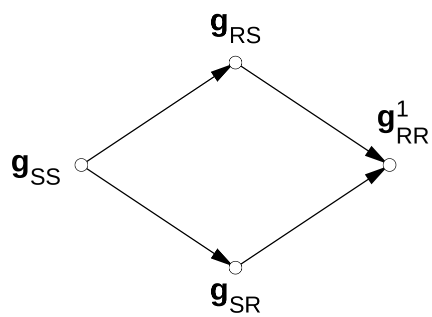

Fig. 10. Partial order between the controllers g_{SS}, g_{SR}, g_{RS} and g^1_{RR} is implied by the physics of sliding and rolling.

Finally, we need to show that the condition (2.2) holds for the regimes SS, SR, and RS. We will again only consider the regime SS, the reasoning for the other two regimes is similar. We first show that the set $E_{SS} = \{w_1 = w_2 = -\frac{v_0}{r},\ \omega = 0\}$ is asymptotically stable for a system as a whole, regardless how the discrete state (dynamic regimes) change. Clearly, this is true for the states w_1 and w_2. Now consider ω as the zero dynamics of the system. As the wheels switch between rolling and sliding, the zero dynamics will change. However, the function V_ω given in Eq. (35) is always decreasing along the trajectories (this was formally shown above for the regime SS, a simple algebraic computation shows it for the other regimes) and a reasoning similar as above shows that V_ω has to converge to 0. According to Lemma A.1, E_{SS} is therefore globally asymptotically stable. Given that E_{SS} is asymptotically stable, we can consider the zero dynamics (ν_x, v_y). Since the dynamics for these two states is nonzero only on M_{SS}, and it was shown above that ν_x and v_y on M_{SS} converge to 0, the zero dynamics (ν_x, v_y) is also asymptotically stable everywhere. We can thus again employ Lemma A.1 to conclude that E^1_{RR} is globally asymptotically stable, regardless how the dynamics switches.

Now take $T > 0$ and assume that a trajectory of the system for $t > T$ evolves on SS for an infinite amount of time. Since E^1_{RR} is asymptotically stable, the trajectory will converge to E^1_{RR}. As noted before, the constraint force on E^1_{RR} is zero, so we can find a neighborhood of E_{RR} in which it is smaller than some $\epsilon > 0$. If any of the wheels starts rolling inside this neighborhood, they will keep rolling for some fixed amount of time, given that the rate of change of the constraint force is bounded (this is the idea of hysteresis, described in Section IV-A). Therefore, the existence of Δ in condition (2.2) is guaranteed. To show that the switch to rolling actually occurs, we need to show that the relative velocity v^i_r of either of the wheels becomes zero (see Eq. 25). But this follows from the fact that in a neighborhood of E^1_{RR}, the rate of change of $|v^i_r|$ is:

$$\frac{d}{dt}|v^i_r| \approx -2\left(\frac{1}{m_b + 2m_w} + \frac{l}{I_b + 2I_v}\right) + 2\frac{C_s}{\sqrt{C_s^2 + 4l^2\omega^2 v_y^2}}\left(\frac{1}{m_b + 2m_w} - \frac{l}{I_b + 2I_v}\right)$$

where

$$C_s = l^2\omega^2 - \nu_x^2 - v_y^2$$

Since this expression is strictly negative, $|v^i_r|$ will decrease at a rate greater than some nonzero constant. It will therefore reach 0 in finite time. This shows that the condition

(2.2) is satisfied and therefore completes the proof of convergence of trajectories to the set E^1_{RR}. In fact, the set will be stable, not only attractive. This is a direct consequence of Corollary III.14, since E^1_{RR} is an asymptotically stable equilibrium set for all four controllers.

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