On the Optimum Dimensioning of Robotic Manipulators

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Abstract. The design of robotic manipulators begins with the dimensioning of the various links to meet performance specifications. However, a methodology for the determination of the manipulator architecture, i.e., the fundamental geometry of the links, regardless of the shape of these, is still lacking. Attempts have been made to apply the classical paradigms of linkage synthesis for motion generation, as in Burmester Theory. The problem with this approach is that it relies on a specific task, described in the form of a discrete set of end-effector poses, which kills the very purpose of using robots, namely, their adaptability to a family of tasks. Another approach relies on the minimization of the condition number of the Jacobian matrix over the architectural parameters and the pose variables of the manipulator. This approach is not trouble-free either, for the matrices involved can have entries of different units, the matrix singular values thus being of disparate dimensions, which prevents the evaluation of the condition number. As a means to solve the dimensional-inhomogeneity problem, the concept of *characteristic length* has been put forth. However, this concept has been slow in finding acceptance within the robotics community, probably because it lacks a direct geometric interpretation. In this paper the concept is revisited and given a simple geometric interpretation. The application of the concept to the design and kinetostatic performance evaluation of serial robots is illustrated with examples.

1. Introduction

In spite of the enormous progress made in the area of robot kinematics since the eighties, a broadly acceptable methodology for the determination of the parameters defining the *fundamental geometry* of both serial and parallel robots is still lacking. Nevertheless, this stage, that we can call link-dimensioning, is key in robot design, for the *fundamental link dimensions* at stake determine all other robot dimensions. Here, we refer to the fundamental geometry of a serial robot as the geometry arising from the relative pose between the axes of the two revolutes—prismatic joints can also be accommodated, but these will be left out of the scope of this paper for conciseness—attached to each link, except for the base and the end-effector. This geometry thus consists of two lines in space at a constant relative pose, defining any of the intermediate robot links. The common normal to the axes can be regarded as a third line intersecting the first two. The distance between the two lines and the *oriented* angle between the two lines are the fundamental link dimensions, which determine the fundamental geometry of the link. Notice that the angle is oriented in the sense that a change of sign of this angle leads to a link with the same fundamental geometry, except for its orientation: one is the mirror image of the other. When all links are coupled to form an open kinematic chain, which constitutes the *skeleton* of the robot, the relative location of the intersection points of the common normals with the revolute axes defines one more fundamental link dimension. All fundamental link dimensions constitute the *architecture parameters* of the robot. A robot architecture does not change as the robot moves. What changes is the relative orientation of the neighboring normals, their distance remaining constant. This orientation is given by an angle that, along with the fundamental dimensions of all the links, constitute the Denavit-Hartenberg parameters of the robot.

Attempts to optimize the fundamental link dimensions of serial manipulators can be traced back to the work of Vinogradov et al. (1971), who introduced the concept of service angle as a figure of merit in robot geometry. The concept was further studied in (Yang and Lai, 1985), while Yoshikawa (1985) introduce *manipulability* as a means to measure the capability of a robot to exert a *twist*, i.e., a velocity of the *operation point* (OP) and an angular velocity of the end-effector (EE). Vijaykumar et al. (1986) proposed guidelines to determine the above dimensions using an optimization approach.

The use of the Jacobian condition number to design a two-link mechanical finger was proposed in (Salisbury and Craig, 1982). The same concept was used to choose an optimum robot posture as an initial guess in computing the robot inverse kinematics (Angeles and Rojas, 1987). Germane to the Jacobian condition number, the concept of conditioning index was put forth in the same context (Angeles and López-Cajún, 1992) as a figure of merit in robot design. This concept was then extended to the design of redundant robots (Angeles, 1995). In the foregoing reference, the characteristic length was introduced as a means to cope with the lack of dimensional homogeneity in the entries of the Jacobian matrix.

The two indices that have been found most suitable for robot design are manipulability (Yoshikawa, 1985) and the condition number (Angeles and Rojas, 1987). Both rely on the singular values of the Jacobian matrix, the former being computed as the product of these, the latter as the ratio of the largest to the smallest one. Manipulability has been found to be too limited in robot design, as it cannot tell a large from a small end-effector, i.e., it is independent of the end-effector operation point, which is the point at which the task is specified (Angeles, 2002). The condition number, in turn, entails the problem of being undefined when the singular values bear distinct dimensions. The introduction of the characteristic length, as a means to allow for a normalization of the Jacobian matrix, i.e., of rendering all its entries dimensionless, solves the problem of lack of dimensional homogeneity. This length is defined in (Tandirci et al., 1992) as that by which the entries of the Jacobian matrix are divided to render it dimensionless and of a minimum condition number at a posture that is found by means of an optimization procedure. This definition of the characteristic length is thus too cumbersome to lend itself to a straightforward geometric interpretation. In this paper we revisit the concept and produce an alternative definition, that does not alter its original significance, but that sheds light on such interpretation.

2. The Concept of Homogeneous Space

Let us define a *homogeneous space* as a Euclidean space whose points bear dimensionless coordinates, the position vector of a point $\overline{P}$ of this space thus being dimensionless, and represented by $\boldsymbol{\rho}$. We will call this point *homogeneous*, to distinguish it from its usual counterpart, whose position vector bears units of length. Note that we will not use Cartesian coordinates of points in this paper, for which reason there should not be confusion between the—dimensionless—coordinates of a homogeneous point and the *homogeneous coordinates* (Angeles, 2002) of a point in the usual sense. Henceforth, we will refer to the three-dimensional Euclidean space as the *usual space* for brevity.

A *homogeneous line* $\mathcal{H}$ is defined as usual, except that its points are all homogeneous, and hence, dimensionless. Line $\mathcal{H}$ is thus fully determined by its direction, given by a unit—and hence, dimensionless—vector $\mathbf{e}$ and a homogeneous point P_H of dimensionless position vector $\boldsymbol{\rho}$ contained in $\mathcal{H}$, as depicted in Fig. 1, where $\overline{O}$ denotes the origin of the homogeneous space.

Line $\mathcal{H}$ can thus be specified by means of vector $\mathbf{e}$ and its dimensionless *moment vector* $\mathbf{n} \equiv \boldsymbol{\rho} \times \mathbf{e}$; both $\mathbf{e}$ and $\mathbf{n}$ will be grouped in the six-dimensional array of *homogeneous Plücker coordinates* $\boldsymbol{\pi}_H$, all of whose entries are dimensionless, namely,

$$\boldsymbol{\pi}_H \equiv \begin{bmatrix} \mathbf{e} \\ \mathbf{n} \end{bmatrix}, \quad \|\mathbf{e}\| = 1, \quad \mathbf{e} \cdot \mathbf{n} = 0 \tag{1}$$

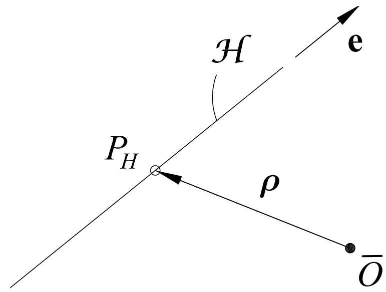


Figure 1. A homogeneous line in homogeneous space

2.1. The Homogeneous Jacobian Matrix of a Serial Manipulator

Consider a set of n homogeneous lines $\mathcal{H}_i$, for $i = 1, 2, \ldots, n$, each line being defined by its unit vector $\mathbf{e}_i$ and its moment $\mathbf{n}_i$. If we regard these lines as the n revolute axes of a *homogeneous manipulator*, then the $6 \times n$ *homogeneous Jacobian matrix* of the manipulator is

$$\mathbf{H} \equiv \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \mathbf{e}_1 \times \boldsymbol{\rho}_1 & \mathbf{e}_2 \times \boldsymbol{\rho}_2 & \cdots & \mathbf{e}_n \times \boldsymbol{\rho}_n \end{bmatrix} \tag{2}$$

in which $-\boldsymbol{\rho}_i$, for $i = 1, 2, \ldots, n$, is defined from a point $\overline{O}_i'$ on $\mathcal{H}_i$ to the *operation point* $\overline{P}$ of the EE of the homogeneous manipulator, as depicted in Fig. 2. In this figure, notice that *any* point on $\mathcal{H}_i$ can be used; however, in order to uniquely define this point in the ensuing optimization procedure, we choose it as that point of $\mathcal{H}_i$ closest to $\overline{P}$, vector $\boldsymbol{\rho}_i$ thus being normal to $\mathbf{e}_i$.

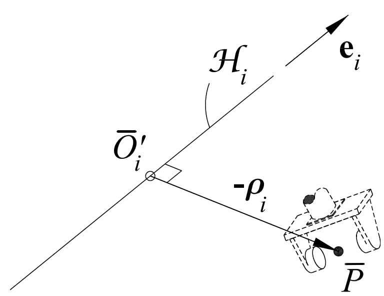


Figure 2. The point $\overline{O}_i'$ on $\mathcal{H}_i$, defined as the point of this line lying closest to the operation point P

In order to compute the homogeneous Jacobian $\mathbf{H}$, we need both a *manipulator architecture* and a *manipulator posture*. The former is given by the set of Denavit-Hartenberg (DH) parameters (Denavit and Hartenberg, 1964) that define the *fundamental geometry* of all the links, i.e., the relative pose between the two revolute axes of a link, and hence, do not change as the manipulator moves: $\{\,\overline{a}_i,\,\overline{b}_i,\,\alpha_i\,\}_1^n$. In this notation, $\overline{a}_i$ is the *homogeneous distance* between $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$, and hence, is a nonnegative, dimensionless real number. In order to define $\overline{b}_i$, we rely on the definition of the DH frames (Denavit and Hartenberg, 1964) $\mathcal{F}_i$, for $i = 1, 2, \ldots, n$, that comprise an origin O_i and coordinate axes X_i, Y_i, Z_i. Since the origins O_i of the DH frames are unambiguously defined by the DH notation, their counterparts $\overline{O}_i$ in homogeneous space need not coincide with points $\overline{O}_i'$ of Fig. 2. As a matter of fact, point $\overline{O}_i$ is fixed on $\mathcal{H}_i$, while point $\overline{O}_i'$ changes its position on this line as the robot changes its posture. Furthermore, we define the *homogeneous DH frames* $\overline{\mathcal{F}}_i$ with origin at $\overline{O}_i$ and axes $\overline{X}_i$, $\overline{Y}_i$, and $\overline{Z}_i$ in homogeneous space. Now, $\overline{b}_i$ can be defined as the $\overline{Z}_i$-coordinate of the intersection of $\overline{Z}_i$ with $\overline{X}_{i+1}$, and hence, is a dimensionless real number that can be positive or negative. Finally, angle α_i, sometimes referred to as the *twist angle* between successive axes, is that

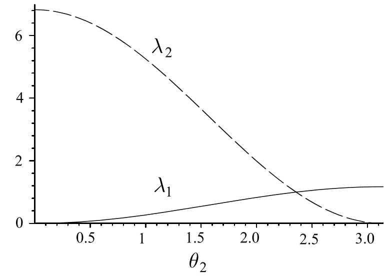


Figure 3. The two singular values of the two-revolute manipulator with $a_2/a_1 = \sqrt{2}/2$ vs. θ_2

between Z_i and Z_{i+1}, and measured positive in the direction of X_{i+1}. Apparently, the twist angle is the same in homogeneous as in the usual space, and hence, we need not distinguish one from the other. Since $\overline{a}_i$ and $\overline{b}_i$ are defined in homogeneous space, they are both dimensionless quantities, and hence, define with the set of twist angles what we can call a *homogeneous manipulator*.

The posture of the manipulator, in turn, is defined by the set of joint variables $\{ \theta_i \}_1^n$, with θ_i measured between axes X_i and X_{i+1}, and defined as positive in the direction of Z_i. Apparently, the joint variables in homogeneous space are identical to those in the usual space, and hence, we need not distinguish one set from the other.

We have thus completely defined the homogeneous Jacobian $\mathbf{H}$, all whose entries are dimensionless quantities. In this vein, then, the *six* singular values of $6 \times n$ $\mathbf{H}$ are all dimensionless, and hence, can be ordered from smallest to largest: $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_6$, where we have assumed $n \geq 6$. The 2-norm condition number (Golub and Van Loan, 1989) $\kappa_2(\mathbf{H})$ of the homogeneous Jacobian $\mathbf{H}$ is, then,

$$\kappa_2(\mathbf{H}) = \frac{\sigma_6}{\sigma_1} \tag{3}$$

The above definition of condition number is simple to state. However, it is rather cumbersome to work with. Indeed, the minimum—as well as the maximum—singular value of a matrix is not an *analytic function* of the matrix *everywhere*, i.e., for *any* real value of its scalar arguments. Here, we recall that a function is analytic at a point if the function admits a series expansion at that point. This requires that the function, first and foremost, have all its derivatives with respect to the argument continuous. To illustrate this point, let us consider the case of a two-revolute planar manipulator, with link-lengths a_1 and a_2. The Jacobian matrix of this manipulator is known to be (Angeles, 2002)

$$\mathbf{J} = a_1 \begin{bmatrix} -s_1 - rs_{12} & -rs_{12} \\ c_1 + rc_{12} & rc_{12} \end{bmatrix}$$

where $r \equiv a_2/a_1$, and, for $i = 1, 2$, $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$. We also define $s_{12} \equiv \sin(\theta_1 + \theta_2)$ and $c_{12} \equiv \cos(\theta_1 + \theta_2)$. Here, θ_i, for $i = 1, 2$, are defined according with the DH notation.

For the special case in which $r = \sqrt{2}/2$, this manipulator is known to have a Jacobian with a condition number of unity (Salisbury and Craig, 1982) when $\theta_2 = 3\pi/4$. The singular values of the above Jacobian, normalized upon dividing them by a_1, vs. θ_2, are displayed in Fig. 3. In this figure, it is apparent that the two singular values trade places as the largest and the smallest at $\theta_2 = 3\pi/4$.

The condition number of the same manipulator, defined as in eq.(3), is bounded from below by unity and unbounded from above. Hence, it is not a good idea to even attempt to plot this quantity. However, its inverse is bounded from below by 0 and from above by unity, as plotted in Fig. 4, which clearly shows a cusp at $\theta_2 = 3\pi/4$. This feature makes the 2-norm condition number extremely cumbersome to work with. As an alternative, we use a definition of the condition number based on the Frobenius norm (Golub and Van Loan, 1989) instead. The Jacobian matrix $\mathbf{H}$ is of $6 \times n$, and hence, rectangular in the case of redundant manipulators, with $n > 6$. For nonredunant

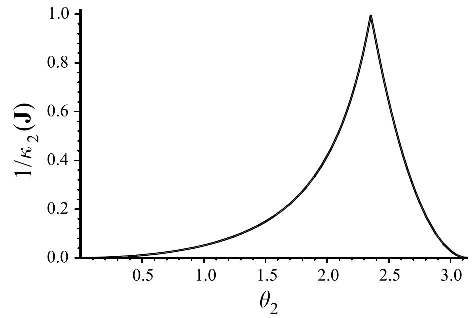


Figure 4. Inverse of condition number $\kappa_2(\mathbf{J})$ vs. θ_2

manipulators, $\mathbf{H}$ is of 6×6, and hence, square. The condition number $\kappa_F(\mathbf{H})$ of a square $\mathbf{H}$, based on the Frobenius norm, is defined as

$$\kappa_F(\mathbf{H}) \equiv \|\mathbf{H}\|_F \|\mathbf{H}^{-1}\|_F \tag{4a}$$

where[1]

$$\|\mathbf{H}\|_F \equiv \sqrt{\frac{1}{6}\mathrm{tr}(\mathbf{H}\mathbf{H}^T)} \equiv \sqrt{\frac{1}{6}\mathrm{tr}(\mathbf{H}^T\mathbf{H})} \tag{4b}$$

in which we have used a weighted norm, as explained in the Appendix, and applied a property of the trace of a product of matrices: The trace of the product is invariant under cyclical permutations of the order of the matrix factors. Additionally, the Frobenius norm of $\mathbf{H}^{-1}$ follows directly from the above expressions. A problem can be detected here when $\mathbf{H}$ is rectangular, for rectangular matrices do not have inverses. Nevertheless, rectangular matrices do have *generalized inverses*. In the case at hand, the *right generalized inverse* $\mathbf{H}^\dagger$ of $6 \times n$ $\mathbf{H}$, with $n > 6$, is defined as (Golub and Van Loan, 1989)

$$\mathbf{H}^\dagger = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T)^{-1} \tag{5}$$

The condition number $\kappa_F(\mathbf{H})$ sought, then, reduces to

$$\kappa_F(\mathbf{H}) = \sqrt{\frac{1}{36}\mathrm{tr}(\mathbf{H}^T\mathbf{H})\mathrm{tr}[(\mathbf{H}^T\mathbf{H})^{-1}]} \equiv \frac{1}{6}\sqrt{\mathrm{tr}(\mathbf{H}^T\mathbf{H})\mathrm{tr}[(\mathbf{H}^T\mathbf{H})^{-1}]} \tag{6}$$

The foregoing expression could have also been obtained without resorting to the generalized inverse. Indeed, if we notice that the singular values of $\mathbf{H}\mathbf{H}^T$ are identical to its eigenvalues and equal to the squares of the singular values of $\mathbf{H}$, then we can write

$$\kappa_F(\mathbf{H}) = \sqrt{\frac{1}{36}\mathrm{tr}(\mathbf{H}\mathbf{H}^T)\mathrm{tr}[(\mathbf{H}\mathbf{H}^T)^{-1}]} \tag{7}$$

which is an expression valid for any $6 \times n$ matrix, with $n \geq 6$.

If we retake our example two-link robot, and compute the condition number of its Jacobian matrix $\mathbf{J}$ based on the Frobenius norm, then we will need $\mathbf{J}^{-1}$, which is

$$\mathbf{J}^{-1} = \frac{1}{a_1 r s_2}\begin{bmatrix} r c_2 & r s_2 \\ -(1 + r c_2) & -r s_2 \end{bmatrix}$$

[1] Nonredundant manipulators with two to five axes require a coefficient of the trace in the definition of the Frobenius norm of 1/2 to 1/5, correspondingly.

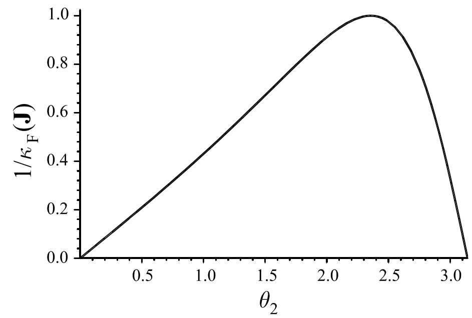


Figure 5. Inverse of condition number $\kappa_F(\mathbf{J})$ vs. θ_2

and $\kappa_F(\mathbf{J})$ is computed as $\kappa = \|\mathbf{J}\|_F \|\mathbf{J}^{-1}\|_F$, where

$$\|\mathbf{J}\|_F = \sqrt{\frac{1}{2}\text{tr}(\mathbf{J}^T\mathbf{J})} = \frac{a_1\sqrt{1+2r^2+2rc_2}}{\sqrt{2}} \tag{8a}$$

Therefore,

$$\|\mathbf{J}^{-1}\|_F = \sqrt{\frac{1}{2}\text{tr}(\mathbf{J}^{-T}\mathbf{J}^{-1})} = \frac{\sqrt{1+2r^2+2rc_2}}{\sqrt{2}\,a_1 r s_2} \tag{8b}$$

Hence,

$$\kappa_F(\mathbf{J}) = \frac{1+2r^2+2rc_2}{2rs_2} \tag{9}$$

A plot of the inverse of the foregoing condition number vs. θ_2 is displayed in Fig. 5.

It is apparent from Fig. 5 that the condition number based on the matrix Frobenius norm is smooth everywhere. This feature makes the condition number thus evaluated quite attractive in applications. For example, this definition allows the use of *gradient methods* to minimize the condition number over architecture parameters and posture variables. For conciseness, however, we leave aside the numerics behind optimization methods in this paper. From eq.(7), the computation of κ_F requires only the inversion of a positive-definite 6×6 matrix. On the contrary, the computation of κ_2, as made apparent by eq.(3), requires an *iterative procedure* to calculate the eigenvalues of $\mathbf{HH}^T$. Since the numerical optimization methods required by the problem at hand fall in the realm of nonlinear programming, which are invariable *iterative*, the use of iterative techniques at each iteration of the optimization procedure itself is strongly recommended against; their use would bring about *nested itertative procedures*.

3. Formulation of the Optimization Problem

In this section we formulate the problem of optimum dimensioning of a n-axis serial manipulator over its architecture parameters $\{\,\overline{a}_i,\, \overline{b}_i,\, \alpha_i\,\}_1^n$ and posture variables $\{\,\theta_i\,\}_1^n$. This problem was solved in (Ranjbaran et al., 1995) for $n = 7$ by means of minimizing the *distance* of the manipulator Jacobian to its isotropic counterpart. The approach we take in this paper is *direct* in that we minimize the condition number itself. Before we embark on the formulation of the optimization problem at hand, it is noteworthy that not all architecture parameters and not all posture variables influence the condition number. Indeed, a change of b_1, or of $\overline{b}_1$ for that matter, with all other architecture parameters unchanged and all joint variables—posture variables—locked, amounts to a pure translation of the manipulator, as if it were a rigid body, along the first revolute axis.

Likewise, a rotation of the manipulator, of fixed architecture, while locking all joints but the first one, amounts to a rotation of the whole manipulator about the first axis, as if it were a rigid body. Furthermore, the nth column of a manipulator Jacobian representing the Plücker coordinates of the nth manipulator axis, this column is not influenced by α_n, which is the angle between Z_n and Z_{n+1}; the latter is an axis attached to the end-effector and defined in terms of the task at hand. Obviously, α_n does not influence the condition number of the manipulator, which does not depend on the task.

Now, since the condition number of a manipulator is an *intrinsic* property of the manipulator, and we have adopted a frame-invariant definition of the condition number—the adopted definition is based on the Frobenius norm, which is frame-invariant—neither rigid-body displacements of the manipulator nor the task influence it. We can now state the result below:

THEOREM 1. *The first joint variable of a serial, n-revolute homogeneous manipulator does not influence the condition number of its homogeneous Jacobian; neither do the architecture parameters* $\bar{b}_1$ *and* α_7.

Notice that the above result cannot be extended to the inhomogeneous Jacobian because no condition number can be associated with it.

Now, if we consider that a n-revolute manipulator has three architecture parameters and one posture variable per joint, its homogeneous Jacobian has a total of $4n$ *design variables* over which the designer can minimize its condition number. Out of these variables, three do not influence the condition number, and hence, we are left with $4n-3$ design variables, which we group in the $(4n-3)$-dimensional vector $\mathbf{x}$, defined as

$$\mathbf{x} \equiv [\,\bar{a}_1 \quad \alpha_1 \quad \bar{a}_2 \quad \bar{b}_2 \quad \alpha_2 \quad \theta_2 \quad \cdots \quad \bar{a}_n \quad \bar{b}_n \quad \theta_n\,]^T \tag{10}$$

However, not all the $4n-3$ components of $\mathbf{x}$ are independent. They are subject to a few constraints, whether directly or indirectly, namely,

$$\|\mathbf{e}_i\| = 1, \quad i = 2, \ldots, n \tag{11a}$$
$$\mathbf{e}_i \cdot \boldsymbol{\rho}_i = 0, \quad i = 1, 2, \ldots, n \tag{11b}$$
$$\bar{a}_i \geq 0, \quad i = 1, 2, \ldots, n \tag{11c}$$

In constraint (11a) we have excluded $\mathbf{e}_1$ because this unit vector is not influenced by any of the design variables. We thus end up with a total of $2n-1$ equality and n inequality constraints. The former are relatively simple to handle in optimum-design problems; the latter, on the contrary, pose some computational difficulties. Although there is a body of knowledge—see, e.g., (Rao, 1996) and the extensive bibliography therein—to deal with inequality constraints, this type of constraints nevertheless can hamper the convergence of any optimization method. Hence, inequality constraints are to be avoided whenever possible. By taking an idea originally proposed by Freudenstein in the context of linkage synthesis (Freudenstein, 1954), and recalling the definition of vector $\mathbf{a}_i$, directed from the origin O_i to the origin O_{i+1} of the DH frames, we can readily dispense with the above inequality constraints. Indeed, the components of $\bar{\mathbf{a}}_i$, the homogeneous counterpart of $\mathbf{a}_i$, in $\overline{\mathcal{F}}_i$-coordinates are (Angeles, 2002)

$$\bar{\mathbf{a}}_i = \begin{bmatrix} \bar{a}_i \cos\theta_i \\ \bar{a}_i \sin\theta_i \\ \bar{b}_i \end{bmatrix}$$

The presence of a solution with $\bar{a}_i < 0$ can be interpreted as indicating a positive $\bar{a}_i$, but with the corresponding θ_i augmented by π. Indeed, upon adding π to an angle, its sine and cosine functions reverse their signs, and hence,

$$\bar{\mathbf{a}}_i = \begin{bmatrix} -\bar{a}_i \cos(\theta_i + \pi) \\ -\bar{a}_i \sin(\theta_i + \pi) \\ \bar{b}_i \end{bmatrix}$$

Table I. Optimum parameters of the *homogeneous manipulator*

Link i	$\bar{a}_i$	$\bar{b}_i$	α_i (deg)	θ_i (deg)
1	0	-	-62.9926	-
2	0.0236	0.0003	-10.2688	35.8567
3	0	0.1732	106.5612	61.7481
4	2.1623	-0.0003	73.7583	116.7073
5	0.0012	-1.8473	55.0046	-24.4698
6	0.0716	3.1639	62.5379	-2.3442
7	1.1590	-1.4464	-	225.5397

Minimum condition number (κ_F) = 1.0000

whence the inequality constraints (11c) can be obviated. In summary, then, we have defined the optimum design problem:

$$\kappa_F(\mathbf{H}) \quad \rightarrow \quad \min_{\mathbf{x}} \tag{12a}$$

subject to

$$\|\mathbf{e}_i\| = 1, \quad i = 2, \ldots, n \tag{12b}$$

$$\mathbf{e}_i \cdot \boldsymbol{\rho}_i = 0, \quad i = 1, 2, \ldots, n \tag{12c}$$

Obviously, if the design task at hand requires specific constraints on link lengths or on twist angles, then these can be incorporated into the foregoing problem to take these constraints into account. In doing so, inequality constraints may still occur.

3.1. Example

We include below one example to illustrate the foregoing concepts.

Example 1: The Design of a Seven-Axis Isotropic Manipulator
In order to solve problem (12a–c) for the case at hand, we resort to the Nelder-Mead simplex method based on function evaluations alone, as implemented in Matlab's Optimization Toolbox (Venkataraman, 2002). As an initial guess, we use the results obtained with the previous approach (Ranjbaran et al., 1995). The results reported by Matlab are recorded in Table I, where all design variables are dimensionless, for they pertain to the *homogeneous manipulator*. In order to obtain the dimensions of the actual manipulator, we multiply $\overline{a}_i$ and $\overline{b}_i$ by the, as yet unknown, characteristic length L of the manipulator. The calculation of L is straightforward, as described below.

We first compute the maximum reach $\overline{R}$ of the homogeneous manipulator, which is done by maximizing the distance of the operation point $\overline{P}$ of the homogeneous manipulator, of position vector $\boldsymbol{\rho}$, from the first revolute axis. Apparently, the first joint variable has no influence on this reach, and hence, can be locked at an arbitrary value of, say 0. The ensuing problem consists then in maximizing the said distance over the remaining six joint variables:

$$\overline{d}^2 \equiv \|\boldsymbol{\rho}_1 - (\boldsymbol{\rho}_1^T \mathbf{e}_1)\mathbf{e}_1\|^2 \quad \equiv \|\mathbf{e}_1 \times \boldsymbol{\rho}_1\|^2 \quad \rightarrow \quad \max_{\boldsymbol{\theta}} \tag{13}$$

subject to no constraints, with the vector of *design variables* $\boldsymbol{\theta}$ defined as

$$\boldsymbol{\theta} = [\,\theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6 \quad \theta_7\,]^T \tag{14}$$

Table II. Joint angles for maximum reach

Link i	θ_i (deg)
1	-
2	319.1493
3	96.6124
4	54.5413
5	64.9203
6	266.3456
7	53.3635
Maximum reach ($\bar{R}$) = 5.9726	

Table III. Optimum a_i and b_i parameters of the actual manipulator with a maximum reach of 877 mm

Link i	a_i mm	b_i mm
1	0	0
2	3.5	0.0
3	0	25.4
4	317.4	0.0
5	0.2	-271.2
6	10.5	464.4
7	1.1590	-212.3

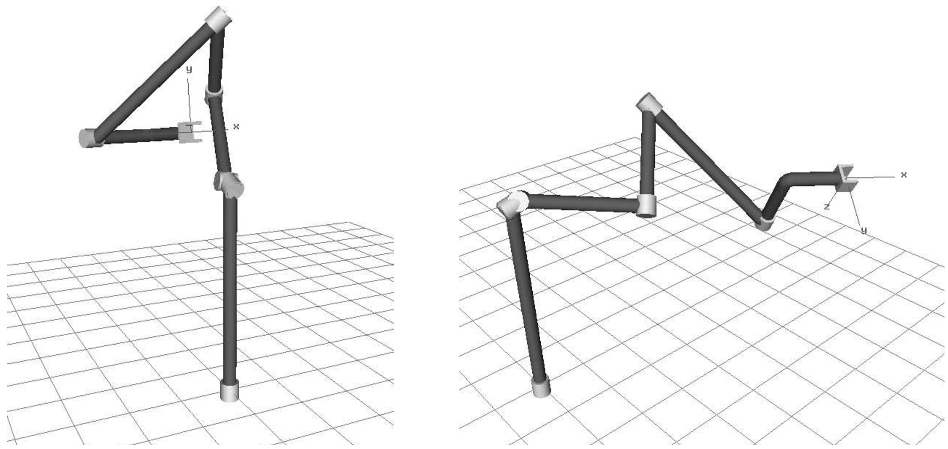
The maximum reach is, thus, $\overline{R} = \max_{\boldsymbol{\theta}}\{\overline{d}\}$. For the isotropic *homogeneous* manipulator, joint-angles for the maximum reach $\overline{R}$, as reported by Matlab, are displayed in Table II.

Finally, if the maximum reach, one of the *manipulator design specifications*, is denoted by R, then the robot characteristic length L is

$$L \equiv \frac{R}{\overline{R}} \tag{15}$$

For example, for a maximum reach of 1866 mm, as in the design reported in (Ranjbaran et al., 1995), $L = 312.4$ mm, while, using the approach of the foregoing paper, L turns out to be about 293 mm. The difference, of slightly over 6%, using as reference the value obtained here, is attributable to the different problem formulations in the two procedures. Moreover, if we want to attain the maximum reach of the Puma 560, which is 877 mm (Angeles, 2002), then the characteristic length of the same manipulator becomes $L = 146.8$ mm. A design with the maximum reach of the Puma 560 thus has the actual Denavit-Hartenberg parameters a_i and b_i of Table III.

The manipulator with the HD parameters of Table III is displayed in Figs. 6(a)& (b), at the isotropic and at the maximum-reach postures, respectively. These figures were produced using RVS (Darcovich et al., 1999).



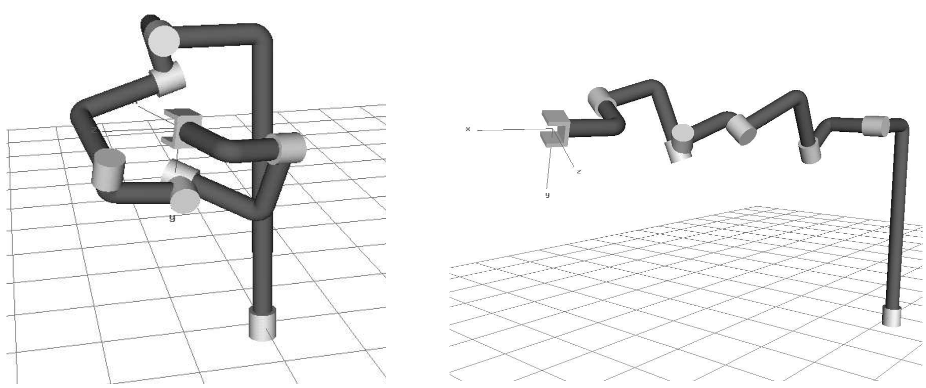
(a) (b)

Figure 6. Displays of optimum manipulator at: (a) isotropic posture; and (b) maximum-reach posture

Apparently, the foregoing optimum manipulator, scale-difference aside, is quite close to that obtained in (Ranjbaran et al., 1995). This should not come as a surprise, for we used the manipulator optimized in the above reference as the initial guess for the optimization procedure.

With a second initial guess, the optimization process is now repeated. This initial guess is based on the parameters of the six-axis DIESTRO robot (Angeles, 2002), designed with an isotropic architecture, to which a seventh link is added with dimensions that follow the pattern of DIESTRO, as recorded in Table IV. The results returned by Matlab are displayed in Table V, which are substantially different from those of Table III. This is an isotropic manipulator whose joint-angles for the maximum reach($\overline{R}$), as reported by Matlab, are displayed in Table VI.

Figures 7(a) & (b) depict geometric models of this manipulator at both the isotropic and the maximum-reach postures, respectively.



(a) (b)

Figure 7. Displays of optimum manipulator at: (a) isotropic posture; and (b) maximum-reach posture

4. The Kinetostatic Performance Evaluation of Serial Robots

The problem formulated in Section 3 can be regarded as *inverse*, in that design specifications are given, and the robot fundamental geometry is required. Here we solve the *direct problem*: Given a

Table IV. Initial guess based on the DIESTRO architecture, with a 7^{th} link added

Link i	$\bar{a}_i$	$\bar{b}_i$	α_i (deg)	θ_i (deg)
1	1	1	90	180
2	1	1	-90	-90
3	1	1	90	90
4	1	1	-90	-90
5	1	1	90	90
6	1	1	-90	0
7	1	1	90	90

Table V. Optimum parameters of the *homogeneous manipulator* obtained with the initial guess of Table IV

Link i	$\bar{a}_i$	$\bar{b}_i$	α_i (deg)	θ_i (deg)
1	1.2695	-	88.3909	-
2	0.8019	0.9903	-96.6278	-105.7176
3	1.2400	1.2726	71.6787	67.0666
4	0.9781	0.3611	-95.4537	-77.1842
5	0.0659	-0.2507	56.3243	82.9464
6	0.9931	1.1546	-127.5050	0.0105
7	1.0711	0.8452	-	106.0311

Minimum condition number (κ_F) = 1.0016

Table VI. Joint angles for optimum manipulator of Table V at the maximum-reach posture

Link i	θ_i (deg)
1	-
2	323.2396
3	333.9036
4	8.6270
5	352.4995
6	0.0014
7	341.3367

Maximum reach ($\bar{R}$) = 7.1264

manipulator architecture, find its characteristic length and, hence, its minimum condition number. The characteristic length of a *given robot* is important because this dimension determines the size of the robot; the minimum condition number of the robot Jacobian determines the design robustness against manufacturing, assembly, and joint-encoder errors. Indeed, the matrix condition number giving an upper bound for the roundoff-error amplification upon solving a system of linear equations whose coefficients are the entries of the given matrix, the smaller the minimum condition number, the less sensitive the robot is to the foregoing errors.

The direct problem is straightforward: Let $a_M \equiv \max_i\{\, a_i \,\}_1^n$, $b_M \equiv \max_i\{\, |b_i| \,\}_2^n$ and $M \equiv \max\{\, a_M, b_M \,\}$. Further, let $\overline{M} \equiv M/L$, where L is the, as yet, unknown characteristic length. Moreover, let

$$\tilde{a}_i \equiv \frac{a_i}{M}, \quad \tilde{b}_i \equiv \frac{b_i}{M}, \quad i = 1, \ldots, n \tag{16a}$$

be a set of *nondimensional* Denavit-Hartenberg parameters of the given robot. Hence,

$$\overline{a}_i = \tilde{a}_i \overline{M}, \quad \overline{b}_i = \tilde{b}_i \overline{M}, \quad i = 1, \ldots, n \tag{16b}$$

Further, $\boldsymbol{\rho}_i$, as introduced in eq.(2), can be expressed as

$$\boldsymbol{\rho}_i = \overline{M}\tilde{\boldsymbol{\rho}}_i, \quad i = 1, \ldots, n \tag{17}$$

where $\tilde{\boldsymbol{\rho}}_i$ is defined exactly as $\boldsymbol{\rho}_i$ was defined for the homogeneous Jacobian $\mathbf{H}$, except that with the nondimensional DH parameters introduced in eq.(16a). Apparently, then, the homogeneous Jacobian can be now expressed as

$$\mathbf{H} \equiv \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \mathbf{e}_1 \times \overline{M}\tilde{\boldsymbol{\rho}}_1 & \mathbf{e}_2 \times \overline{M}\tilde{\boldsymbol{\rho}}_2 & \cdots & \mathbf{e}_n \times \overline{M}\tilde{\boldsymbol{\rho}}_n \end{bmatrix} \tag{18}$$

Hence, to find the characteristic length L, all we need is find the value of $\overline{M}$ that will render, along with suitable values of the $n-1$ joint variables, the condition number of the homogeneous Jacobian $\mathbf{H}$ a minimum. Let all these optimizing variables be grouped in the new design vector $\tilde{\mathbf{x}}$, namely,

$$\tilde{\mathbf{x}} \equiv [\, \overline{M} \quad \theta_2 \quad \cdots \quad \theta_n \,]^T \tag{19}$$

The value of vector $\tilde{\mathbf{x}}$ is thus found as the solution to an optimization problem:

$$\kappa_F^2(\mathbf{H}) \quad \rightarrow \quad \min_{\tilde{\mathbf{x}}} \tag{20}$$

subject to no constraints, except for $\overline{M} > 0$, which is readily implementable. Indeed, given the definition of $\kappa_F(\mathbf{H})$ in eq.(7), $\kappa_F^2(\mathbf{H})$ is apparently a quadratic function of $\overline{M}$ and of its inverse, the outcome being that $\kappa_F^2(\mathbf{H})$ is an even function of $\overline{M}$, i.e.,

$$\kappa_F^2(\overline{M}) = \kappa_F^2(-\overline{M}) \tag{21}$$

and hence, if $-\overline{M}$ is a solution to the optimization problem (20), then so is $\overline{M}$, and we need not worry about enforcing the sign constraint on $\overline{M}$.

Example 2: Performance Evaluation of an Industrial Robot
In order to illustrate the foregoing procedure, we calculate the characteristic length and the minimum condition number of the homogeneous Jacobian of the Fanuc Arc Mate manipulator whose DH parameters are included in Table VII. Note that these items were computed in (Angeles, 2002) upon minimization of the condition number based on the 2-norm. The results obtained in that reference are $L = 357.3$ mm and $\kappa_2 = 2.589$. The condition number based on Frobenius norm is $\kappa_F = 1.3040$ at this posture.

Table VII. DH Parameters of the Fanuc Arc Mate Manipulator

Link i	a_i (mm)	b_i (mm)	α_i (deg)	θ_i (deg)
1	200	810	90	θ_1
2	600	0	0	θ_2
3	130	30	90	θ_3
4	0	550	90	θ_4
5	0	100	90	θ_5
6	0	100	0	θ_6
Minimum condition number (κ_F) = 1.3040				

The optimum vector $\tilde{\mathbf{x}}_{opt}$ obtained is reported below:

$$\tilde{\mathbf{x}}_{opt} = [\,1.7083 \quad 22.60 \quad -51.13 \quad -20.07 \quad -88.00\,]^T$$

where the last joint angle θ_6 was was not included as it does not effect the condition number for this particular architecture.

The characteristic length is thus equal to

$$L = \frac{M}{\overline{\overline{M}}} = \frac{600}{1.7083} = 351.23 \text{ mm}$$

with the condition number $\kappa_F = 1.2717$. Note that the condition number based on 2-norm at this posture is $\kappa_2 = 2.7254$.

5. Conclusions

The fundamental geometry of a serial robot was defined here, with a methodology proposed as to how to determine it at the preliminary design stage. In order to cope with the lack of dimensional homogeneity in the entries of the robot Jacobian, the characteristic length was recalled. In doing so, a definition of this length was proposed, that keeps the significance of previous definitions, but that bears a straightforward geometric interpretation. The robot is thus designed in homogeneous space, and then mapped into the real Euclidean space by means of the characteristic length, which is determined from the maximum reach, that is usually prescribed when designing a robot.

The design procedure relies on an unconstrained optimization scheme. This procedure was illustrated with the design of a seven-axis redundant robot, whose objective function is non-quadratic and nonconvex, the problem thus admitting many local optima. By starting the optimization procedure from two distinct initial guesses, two distinct isotropic robots were obtained, each with DH parameters close to those of the initial guess. Therefore, if certain DH parameters are *desired*, then these can be enforced via the initial guess. This problem was termed *inverse*.

The *direct problem* was also addressed here. For a robot with $n \geq 6$ revolute axes, of a given architecture, i.e., of given DH parameters, we showed that both the characteristic length and the minimum condition number can be found via the solution of an unconstrained minimization problem. The approach proposed here was illustrated with the example of a Fanuc Arc Mate robot.

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Appendix: The Weighted Frobenius Norm

The weighted Frobenius norm of a $n \times n$ matrix $\mathbf{A}$ is defined as

$$\|\mathbf{A}\|_F = \sqrt{\mathbf{A}\mathbf{W}\mathbf{A}^T} \tag{22}$$

in which $\mathbf{W}$ is a positive-definite *weighting matrix*, which is introduced with the purpose of *normalizing* the norm according to the need of a specific context. It is desired, for example, to have the weighted Frobenius norm of the $n \times n$ identity matrix $\mathbf{1}$ defined as unity, which then requires that $\mathbf{W}$ be defined as

$$\mathbf{W} = \frac{1}{n}\mathbf{1} \tag{23}$$