Computer Science Department

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Introduction to Calculus of Variations

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Calculus of Variations

Generalization of Calculus that seeks to find the path, curve, surface, etc., for which a given Functional has a minimum or maximum.

Goal: find extrema values of integrals of the form

$$\int F(u, u_x)dx$$

It has an extremum only if the Euler-Lagrange Differential Equation is satisfied,

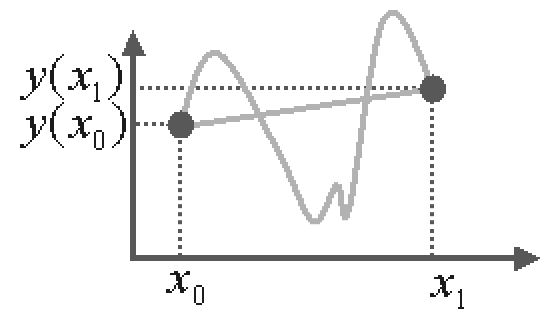
$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right)F(u, u_x) = 0$$

Calculus of Variations

Example: Find the shape of the curve {x,y(x)} with shortest length:

$$\int_{x_0}^{x_1} \sqrt{1+ y_x{}^2}\, dx,$$

given *y(x0),y(x1)*



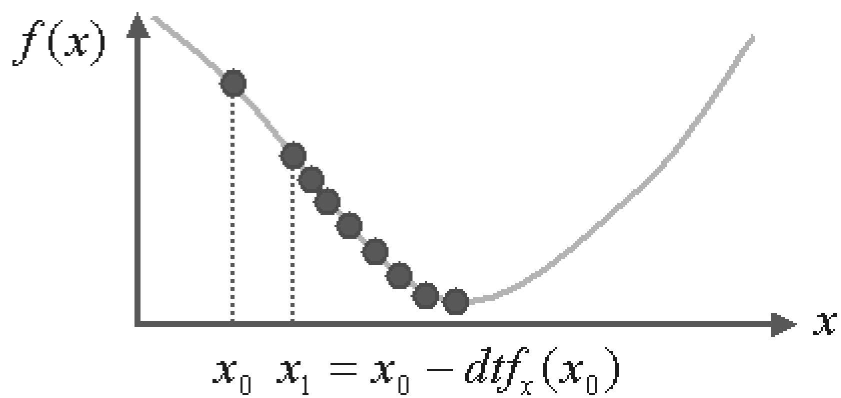
Solution: a differential equation that *y(x)* must satisfy,

$$\frac{y_{xx}}{\left(1+ y_x{}^2\right)^{3/2}} = 0 \quad \Rightarrow \quad y_x = a \Rightarrow \quad y(x) = ax + b$$

Extrema points in calculus

$$\forall\, \eta : \lim_{\varepsilon \to 0}\left(\frac{df(x+\varepsilon\eta)}{d\varepsilon} \right) = 0 \Leftrightarrow \forall\, \eta : f_x(x)\eta = 0 \Leftrightarrow f_x(x) = 0$$

Gradient descent process $x_t = -f_x$



Calculus of variations

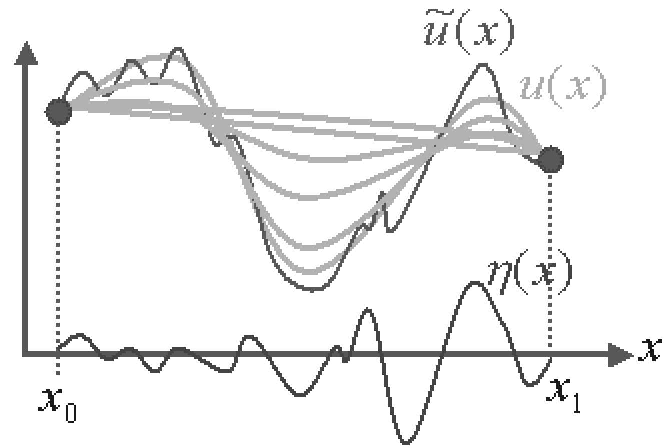
$$E(u(x)) = \int F(u, u_x)\,dx$$

$$\tilde{u}(x) = u(x) + \varepsilon\eta(x)$$

$$\forall\, \eta(x) : \lim_{\varepsilon \to 0}\left(\frac{d}{d\varepsilon}\int F(\tilde{u}, \tilde{u}_x)\,dx\right) \stackrel{?}{=} 0$$

⇩

$$\frac{\delta E(u)}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right) F(u, u_x)$$



Gradient descent process $u_t = -\dfrac{\delta E(u)}{\delta u}$

Euler Lagrange Equation

Proof. for fixed *u(x0),u(x1)* :

$$\int \frac{d}{d\varepsilon} F(\tilde{u}, \tilde{u}_x) dx = \int (F_{\tilde{u}} \tilde{u}_\varepsilon + F_{\tilde{u}} \tilde{u}_{x\varepsilon}) dx = \int (F_{\tilde{u}} \eta + F_{\tilde{u}} \eta_x) dx$$

$$= \int F_{\tilde{u}} \eta dx + F_{\tilde{u}} \eta \Big|_{x_0}^{x_1} - \int \eta \frac{d}{dx} (F_{\tilde{u}}) dx$$

$$= \int \left(F_{\tilde{u}} - \frac{d}{dx} (F_{\tilde{u}}) \right) \eta dx$$

Thus the Euler Lagrange equation is

$$\left(\frac{\partial}{\partial u} - \frac{d}{dx} \frac{\partial}{\partial u_x} \right) F(u, u_x) = 0$$