Boost.Numeric.Odeint

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Getting started

Overview

Caution

Boost.Numeric.Odeint is not an official boost library!

odeint is a library for solving initial value problems (IVP) of ordinary differential equations. Mathematically, these problems are formulated as follows:

\[ x'(t) = f(x,t), \quad x(0) = x_0. \]

\( x \) and \( f \) can be vectors and the solution is some function \( x(t) \) fulfilling both equations above. In the following we will refer to \( x'(t) \) also as \( \frac{dx}{dt} \), which is also our notation for the derivative in the source code.

Ordinary differential equations occur nearly everywhere in natural sciences. For example, the whole Newtonian mechanics are described by second order differential equations. Be sure, you will find them in every discipline. They also occur if partial differential equations (PDEs) are discretized. Then, a system of coupled ordinary differential occurs, sometimes also referred as lattices ODEs.

Numerical approximations for the solution \( x(t) \) are calculated iteratively. The easiest algorithm is the Euler scheme, where starting at \( x(0) \) one finds \( x(dt) = x(0) + dt f(x(0),0) \). Now one can use \( x(dt) \) and obtain \( x(2dt) \) in a similar way and so on. The Euler method is of order 1, that means the error at each step is \( \sim dt^2 \). This is, of course, not very satisfying, which is why the Euler method is rarely used for real life problems and serves just as illustrative example.

The main focus of odeint is to provide numerical methods implemented in a way where the algorithm is completely independent on the data structure used to represent the state \( x \). In doing so, odeint is applicable for a broad variety of situations and it can be used with many other libraries. Besides the usual case where the state is defined as a `std::vector` or a `boost::array`, we provide native support for the following libraries:

- Boost.uBLAS
- Thrust, making odeint naturally running on CUDA devices
- gsl_vector for compatibility with the many numerical function in the GSL
- Boost.Range
- Boost.Fusion (the state type can be a fusion vector)
- Boost.Units
- Intel Math Kernel Library for maximum performance
- VexCL for OpenCL
- Boost.Graph (still experimentally)

In odeint, the following algorithms are implemented:
Table 1. Stepper Algorithms

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<tr>
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<th>Class</th>
<th>Concept</th>
<th>System Concept</th>
<th>Order</th>
<th>Error Estimation</th>
<th>Dense Output</th>
<th>Internal State</th>
<th>Remarks</th>
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<td>Explicit Euler</td>
<td>euler</td>
<td>Dense Output</td>
<td>Stepper System</td>
<td>1</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Very simple, only for demonstrating purpose</td>
</tr>
<tr>
<td>Modified Midpoint</td>
<td>modified_midpoint</td>
<td>Stepper System</td>
<td>System</td>
<td></td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Used in Bulirsch-Stoer implementation</td>
</tr>
<tr>
<td>Runge-Kutta 4</td>
<td>runge_kutta4</td>
<td>Stepper System</td>
<td>System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>The classical Runge-Kutta scheme, good general scheme without error control</td>
</tr>
<tr>
<td>Cash-Karp</td>
<td>runge_kutta_cash</td>
<td>Error Stepper</td>
<td>System</td>
<td>5</td>
<td>Yes (4)</td>
<td>No</td>
<td>No</td>
<td>Good general scheme with error estimation, to be used in controlled_error_stepper</td>
</tr>
<tr>
<td>Dormand-Prince 5</td>
<td>runge_kutta_dp5</td>
<td>Error Stepper</td>
<td>System</td>
<td>5</td>
<td>Yes (4)</td>
<td>Yes</td>
<td>Yes</td>
<td>Standard method with error control and dense output, to be used in controlled_error_stepper and in dense_output_controlled_explicit_fsal.</td>
</tr>
<tr>
<td>Fehlberg 78</td>
<td>runge_kutta_fehlberg78</td>
<td>Error Stepper</td>
<td>System</td>
<td>8</td>
<td>Yes (7)</td>
<td>No</td>
<td>No</td>
<td>Good high order method with error estimation, to be used in controlled_error_stepper.</td>
</tr>
<tr>
<td>Algorithm</td>
<td>Class</td>
<td>Concept</td>
<td>System</td>
<td>Order</td>
<td>Error Estimation</td>
<td>Dense Output</td>
<td>Internal State</td>
<td>Remarks</td>
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<td>-----------------------------</td>
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<td>Adams Bashforth</td>
<td>adams_bashforth</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Multistep method</td>
</tr>
<tr>
<td>Adams Moulton</td>
<td>adams_moulton</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Multistep method</td>
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<tr>
<td>Adams Bashforth Moulton</td>
<td>adams_bashforth_moulton</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Combined multistep method</td>
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<td>controlled_runge_kutta</td>
<td>Controlled Stepper</td>
<td>System</td>
<td>depends</td>
<td>Yes</td>
<td>No</td>
<td>depends</td>
<td>Error control for Error Stepper. Requires an Error Stepper from above. Order depends on the given Error Stepper.</td>
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<td>dense_output_runge_kutta</td>
<td>Dense Output Stepper</td>
<td>System</td>
<td>depends</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Dense output for Stepper and Error Stepper from above if they provide dense output functionality (like euler and runge_kutta). Order depends on the given stepper.</td>
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<td>bulirsch_stoer</td>
<td>Controlled Stepper</td>
<td>System</td>
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<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Stepper with step size and order control. Very good if high precision is required.</td>
</tr>
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<td>Class</td>
<td>Concept</td>
<td>System Concept</td>
<td>Order</td>
<td>Error Estimation</td>
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<td>Bulirsch-Stoer Dense Output</td>
<td>bu-lirsch_sto-er_dense_out</td>
<td>Dense Output Stepper</td>
<td>System</td>
<td>variable</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Step with step size and order control as well as dense output. Very good if high precision and dense output is required.</td>
</tr>
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<td>Implicit Euler</td>
<td>implicit_euler</td>
<td>Stepper</td>
<td>Implicit System</td>
<td>1</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Basic implicit routine. Requires the Jacobian. Works only with BoostuBLAS vectors as state types.</td>
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<td>Rosenbrock 4</td>
<td>rosen-brock4</td>
<td>Error Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Good for stiff systems. Works only with BoostuBLAS vectors as state types.</td>
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<td>rosen-brock4_controller</td>
<td>Controlled Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Rosenbrock 4 with error control. Works only with BoostuBLAS vectors as state types.</td>
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<tr>
<td>Dense Output Rosenbrock 4</td>
<td>rosen-brock4dense_output</td>
<td>Dense Output Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Controlled Rosenbrock 4 with dense output. Works only with BoostuBLAS vectors as state types.</td>
</tr>
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<td>Class</td>
<td>Concept</td>
<td>System Concept</td>
<td>Order</td>
<td>Error Estimation</td>
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<td>Internal State</td>
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<td>symplectic_euler</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>1</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Basic symplectic solver for separable Hamiltonian system</td>
</tr>
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<td>Symplectic RKN McLachlan</td>
<td>symplectic_rkn_mclachlan</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Symplectic solver for separable Hamiltonian system with 6 stages and order 4</td>
</tr>
<tr>
<td>Symplectic RKN McLachlan</td>
<td>symplectic_rkn_mclachlan</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Symplectic solver with 5 stages and order 4, can be used with arbitrary precision types</td>
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**Usage, Compilation, Headers**

odeint is a header-only library, no linking against pre-compiled code is required. It can be included by

```plaintext
#include <boost/numeric/odeint.hpp>
```

which includes all headers of the library. All functions and classes from odeint live in the namespace

```plaintext
using namespace boost::numeric::odeint;
```

It is also possible to include only parts of the library. This is the recommended way since it saves a lot of compilation time.

- #include <boost/numeric/odeint/stepper/XYZ.hpp> - the include path for all steppers, XYZ is a placeholder for a stepper.
- #include <boost/numeric/odeint/algebra/XYZ.hpp> - all algebras.
- #include <boost/numeric/odeint/util/XYZ.hpp> - the utility functions like is_resizeable, same_size, or resize.
- #include <boost/numeric/odeint/integrate/XYZ.hpp> - the integrate routines.
- #include <boost/numeric/odeint/iterator/XYZ.hpp> - the range and iterator functions.
- #include <boost/numeric/odeint/external/XYZ.hpp> - any binders to external libraries.
Short Example

Imaging, you want to numerically integrate a harmonic oscillator with friction. The equations of motion are given by \( x'' = -x + \gamma x' \). Odeint only deals with first order ODEs that have no higher derivatives than \( x' \) involved. However, any higher order ODE can be transformed to a system of first order ODEs by introducing the new variables \( q = x \) and \( p = x' \) such that \( w = (q, p) \). To apply numerical integration one first has to design the right hand side of the equation \( w' = f(w) = (p, -q + \gamma p) \):

```cpp
/* The type of container used to hold the state vector */
typedef std::vector< double > state_type;

const double gam = 0.15;

/* The rhs of \( x' = f(x) \) */
void harmonic_oscillator( const state_type &x , state_type &dxdt , const double /* t */ )
{
    dxdt[0] = x[1];
    dxdt[1] = -x[0] - gam*x[1];
}
```

Here we chose `vector<double>` as the state type, but others are also possible, for example `boost::array<double,2>`. odeint is designed in such a way that you can easily use your own state types. Next, the ODE is defined which is in this case a simple function calculating \( f(x) \). The parameter signature of this function is crucial: the integration methods will always call them in the form \( f(x, \ dxdt, \ t) \) (there are exceptions for some special routines). So, even if there is no explicit time dependence, one has to define \( t \) as a function parameter.

Now, we have to define the initial state from which the integration should start:

```cpp
state_type x(2);
x[0] = 1.0; // start at x=1.0, p=0.0
x[1] = 0.0;
```

For the integration itself we'll use the `integrate` function, which is a convenient way to get quick results. It is based on the error-controlled `runge_kutta54_cash_karp` stepper (5th order) and uses adaptive step-size.

```cpp
size_t steps = integrate( harmonic_oscillator ,
                          x ,  0.0 , 10.0 , 0.1 );
```

The `integrate` function expects as parameters the rhs of the ode as defined above, the initial state \( x \), the start-and end-time of the integration as well as the initial time step=size. Note, that `integrate` uses an adaptive step-size during the integration steps so the time points will not be equally spaced. The integration returns the number of steps that were applied and updates \( x \) which is set to the approximate solution of the ODE at the end of integration.

It is also possible to represent the ode system as a class. The rhs must then be implemented as a functor - a class with an overloaded function call operator:
/* The rhs of x' = f(x) defined as a class */

```cpp
class harm_osc {
  double m_gam;

public:
  harm_osc( double gam ) : m_gam(gam) { }

  void operator() ( const state_type &x , state_type &dxdt , const double /* t */ )
  {
    dxdt[0] = x[1];
    dxdt[1] = -x[0] - m_gam*x[1];
  }
};
```

which can be used via

```cpp
harm_osc ho(0.15);
steps = integrate( ho ,
                 x , 0.0 , 10.0 , 0.1 );
```

In order to observe the solution during the integration steps all you have to do is to provide a reasonable observer. An example is

```cpp
struct push_back_state_and_time
{
  std::vector< state_type >& m_states;
  std::vector< double >& m_times;

  push_back_state_and_time( std::vector< state_type > &states , std::vector< double > &times ) : m_states( states ) , m_times( times ) { }

  void operator()( const state_type &x , double t )
  {
    m_states.push_back( x );
    m_times.push_back( t );
  }
};
```

which stores the intermediate steps in a container. Note, the argument structure of the ()-operator: odeint calls the observer exactly in this way, providing the current state and time. Now, you only have to pass this container to the integration function:

```cpp
vector<state_type> x_vec;
vector<double> times;

steps = integrate( harmonic_oscillator ,
                 x , 0.0 , 10.0 , 0.1 ,
                 push_back_state_and_time( x_vec , times ) );

/* output */
for( size_t i=0; i<=steps; i++ )
{
  cout << times[i] << '\t' << x_vec[i][0] << '\t' << x_vec[i][1] << '\n';
}
```

That is all. You can use functional libraries like Boost.Lambda or Boost.Phoenix to ease the creation of observer functions.

The full cpp file for this example can be found here: harmonic_oscillator.cpp
**Tutorial**

**Harmonic oscillator**

**Define the ODE**

First of all, you have to specify the data type that represents a state $x$ of your system. Mathematically, this usually is an $n$-dimensional vector with real numbers or complex numbers as scalar objects. For odeint the most natural way is to use `vector< double >` or `vector< complex< double > >` to represent the system state. However, odeint can deal with other container types as well, e.g. `boost::array< double , N >`, as long as it fulfills some requirements defined below.

To integrate a differential equation numerically, one also has to define the rhs of the equation $x' = f(x)$. In odeint you supply this function in terms of an object that implements the ()-operator with a certain parameter structure. Hence, the straightforward way would be to just define a function, e.g:

```cpp
/* The type of container used to hold the state vector */
typedef std::vector< double > state_type;

const double gam = 0.15;

/* The rhs of x' = f(x) */
void harmonic_oscillator( const state_type &x , state_type &dxdt , const double /* t */ )
{
    dxdt[0] = x[1];
    dxdt[1] = -x[0] - gam*x[1];
}
```

The parameters of the function must follow the example above where $x$ is the current state, here a two-component vector containing position $q$ and momentum $p$ of the oscillator, $dxdt$ is the derivative $x'$ and should be filled by the function with $f(x)$, and $t$ is the current time. Note that in this example $t$ is not required to calculate $f$, however odeint expects the function signature to have exactly three parameters (there are exception, discussed later).

A more sophisticated approach is to implement the system as a class where the rhs function is defined as the ()-operator of the class with the same parameter structure as above:

```cpp
/* The rhs of x' = f(x) defined as a class */
class harm_osc {
    double m_gam;

public:
    harm_osc( double gam ) : m_gam(gam) { }

    void operator() ( const state_type &x , state_type &dxdt , const double /* t */ )
    {
        dxdt[0] = x[1];
        dxdt[1] = -x[0] - m_gam*x[1];
    }
};
```

odeint can deal with instances of such classes instead of pure functions which allows for cleaner code.

**Stepper Types**

Numerical integration works iteratively, that means you start at a state $x(t)$ and perform a time-step of length $dt$ to obtain the approximate state $x(t+dt)$. There exist many different methods to perform such a time-step each of which has a certain order $q$. If the order
of a method is $q$ than it is accurate up to term $\sim dt^{q+1}$, that means the error in $x$ made by such a step is $\sim dt^{q+1}$. odeint provides several steppers of different orders, see Stepper overview.

Some of steppers in the table above are special: Some need the Jacobian of the ODE, others are constructed for special ODE-systems like Hamiltonian systems. We will show typical examples and use-cases in this tutorial and which kind of steppers should be applied.

### Integration with Constant Step Size

The basic stepper just performs one time-step and doesn't give you any information about the error that was made (except that you know it is of order $q+1$). Such steppers are used with constant step size that should be chosen small enough to have reasonable small errors. However, you should apply some sort of validity check of your results (like observing conserved quantities) because you have no other control of the error. The following example defines a basic stepper based on the classical Runge-Kutta scheme of 4th order. The declaration of the stepper requires the state type as template parameter. The integration can now be done by using the `integrate_const()` function from odeint:

```cpp
class runge_kutta4< state_type > stepper;
integrate_const( stepper, harmonic_oscillator, x, 0.0, 10.0, 0.01 );
```

This call integrates the system defined by `harmonic_oscillator` using the RK4 method from $t=0$ to $10$ with a step-size $dt=0.01$ and the initial condition given in $x$. The result, $x(t=10)$ is stored in $x$ (in-place). Each stepper defines a `do_step` method which can also be used directly. So, you write down the above example as

```cpp
const double dt = 0.01;
for( double t=0.0 ; t<10.0 ; t+= dt )
    stepper.do_step( harmonic_oscillator, x, t, dt );
```

#### Tip

If you have a C++11 enabled compiler you can easily use lambdas to create the system function:

```cpp
runge_kutta4< state_type > stepper;
integrate_const( stepper, []( const state_type &x , state_type &dxdt , double t ) {
    dxdt[0] = x[1]; dxdt[1] = -x[0] - gam*x[1];
}, x, 0.0, 10.0, 0.01 );
```

### Integration with Adaptive Step Size

To improve the numerical results and additionally minimize the computational effort, the application of a step size control is advisable. Step size control is realized via stepper algorithms that additionally provide an error estimation of the applied step. odeint provides a number of such ErrorSteppers and we will show their usage on the example of `explicit_error_rk54_ck` a 5th order Runge-Kutta method with 4th order error estimation and coefficients introduced by Cash and Karp.

```cpp
typedef runge_kutta_cash_karp54< state_type > error_stepper_type;
```

Given the error stepper, one still needs an instance that checks the error and adjusts the step size accordingly. In odeint, this is done by ControlledSteppers. For the `runge_kutta_cash_karp54` stepper a `controlled_runge_kutta` stepper exists which can be used via

```cpp
typedef controlled_runge_kutta< error_stepper_type > controlled_stepper_type;
controlled_stepper_type controlled_stepper;
integrate_adaptive( controlled_stepper, harmonic_oscillator, x, 0.0, 10.0, 0.01 );
```
As above, this integrates the system defined by harmonic_oscillator, but now using an adaptive step size method based on the Runge-Kutta Cash-Karp 54 scheme from $t=0$ to $10$ with an initial step size of $dt=0.01$ (will be adjusted) and the initial condition given in $x$. The result, $x(t=10)$, will also be stored in $x$ (in-place).

In the above example an error stepper is nested in a controlled stepper. This is a nice technique; however one drawback is that one always needs to define both steppers. One could also write the instantiation of the controlled stepper into the call of the integrate function but a complete knowledge of the underlying stepper types is still necessary. Another point is, that the error tolerances for the step size control are not easily included into the controlled stepper. Both issues can be solved by using make_controlled:

```c++
integrate_adaptive( make_controlled<error_stepper_type>(1.0e-10, 1.0e-6),
harmonic_oscillator, x, 0.0, 10.0, 0.01);
```

make_controlled can be used with many of the steppers of odeint. The first parameter is the absolute error tolerance $\text{eps\_abs}$ and the second is the relative error tolerance $\text{eps\_rel}$ which is used during the integration. The template parameter determines from which error stepper a controlled stepper should be instantiated. An alternative syntax of make_controlled is

```c++
integrate_adaptive( make_controlled(1.0e-10, 1.0e-6, error_stepper_type()),
harmonic_oscillator, x, 0.0, 10.0, 0.01);
```

For the Runge-Kutta controller the error made during one step is compared with $\text{eps\_abs} + \text{eps\_rel} \times (a_x \times |x| + a_{dxdt} \times dt \times |dxdt|)$. If the error is smaller than this value the current step is accepted, otherwise it is rejected and the step size is decreased. Note, that the step size is also increased if the error gets too small compared to the rhs of the above relation. The full instantiation of the controlled_runge_kutta with all parameters is therefore

```c++
double abs_err = 1.0e-10, rel_err = 1.0e-6, a_x = 1.0, a_dxdt = 1.0;
controlled_stepper_type controlled_stepper(
    default_error_checker<double>({abs_err, rel_err, a_x, a_dxdt}));
integrate_adaptive( controlled_stepper, harmonic_oscillator, x, 0.0, 10.0, 0.01);
```

When using make_controlled the parameter $a_x$ and $a_{dxdt}$ are used with their standard values of 1.

In the tables below, one can find all steppers which are working with make_controlled and make_dense_output which is the analog for the dense output steppers.

### Table 2. Generation functions make_controlled(\textbf{abs\_error} , \textbf{rel\_error} , \textbf{stepper} )

<table>
<thead>
<tr>
<th>Stepper</th>
<th>Result of make_controlled</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>runge_kutta_cash_karp54</td>
<td>controlled_runge_kutta&lt;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>runge_kutta_cash_karp54 , default_error_checker&lt;...&gt; &gt;</td>
<td>$a_x=1$, $a_{dxdt}=1$</td>
</tr>
<tr>
<td>runge_kutta_fehlberg78</td>
<td>controlled_runge_kutta&lt;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>runge_kutta_fehlberg78 , default_error_checker&lt;...&gt; &gt;</td>
<td>$a_x=1$, $a_{dxdt}=1$</td>
</tr>
<tr>
<td>runge_kutta_dopri5</td>
<td>controlled_runge_kutta&lt;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>runge_kutta_dopri5 , default_error_checker&lt;...&gt; &gt;</td>
<td>$a_x=1$, $a_{dxdt}=1$</td>
</tr>
<tr>
<td>rosenbrock4</td>
<td>rosenbrock4_controlled&lt;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>rosenbrock4 &gt;</td>
<td>-</td>
</tr>
</tbody>
</table>
Make the dense output functions: `make_dense_output( abs_error , rel_error , stepper )`

<table>
<thead>
<tr>
<th>Stepper</th>
<th>Result of make_dense_output</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>runge_kutta_dopri5</code></td>
<td><code>dense_output_runge_kutta&lt; controlled_runge_kutta&lt; runge_kutta_dopri5 , default_error_checker&lt;&gt; &gt; &gt;</code></td>
<td><code>a_x=l, aહ_t=1</code></td>
</tr>
<tr>
<td><code>rosenbrock4</code></td>
<td><code>rosenbrock4_dense_output&lt; rosenbrock4_controller&lt; rosenbrock4 &gt; &gt;</code></td>
<td>-</td>
</tr>
</tbody>
</table>

When using `make_controlled` or `make_dense_output` one should be aware which exact type is used and how the step size control works.

The full source file for this example can be found here: `harmonic_oscillator.cpp`

**Solar system**

**Gravitation and energy conservation**

The next example in this tutorial is a simulation of the outer solar system, consisting of the sun, Jupiter, Saturn, Uranus, Neptune and Pluto.

Each planet and of course the sun will be represented by mass points. The interaction force between each object is the gravitational force which can be written as

\[
F_{ij} = -\gamma m_i m_j (q_i - q_j) / |q_i - q_j|^3
\]
where \( \gamma \) is the gravitational constant, \( m_i \) and \( m_j \) are the masses and \( q_i \) and \( q_j \) are the locations of the two objects. The equations of motion are then

\[
\begin{align*}
\frac{dq_i}{dt} &= p_i \\
\frac{dp_i}{dt} &= \frac{1}{m_i} \sum_j F_{ij}
\end{align*}
\]

where \( p_i \) is the momenta of object \( i \). The equations of motion can also be derived from the Hamiltonian

\[
H = \sum_i \frac{p_i^2}{2m_i} + \sum_j V(q_i, q_j)
\]

with the interaction potential \( V(q_i, q_j) \). The Hamiltonian equations give the equations of motion

\[
\begin{align*}
\frac{dq_i}{dt} &= \frac{dH}{dp_i} \\
\frac{dp_i}{dt} &= -\frac{dH}{dq_i}
\end{align*}
\]

In time independent Hamiltonian system the energy and the phase space volume are conserved and special integration methods have to be applied in order to ensure these conservation laws. The odeint library provides classes for separable Hamiltonian systems, which can be written in the form \( H = \sum_i \frac{p_i^2}{2m_i} + H_q(q) \), where \( H_q(q) \) only depends on the coordinates. Although this functional form might look a bit arbitrary, it covers nearly all classical mechanical systems with inertia and without dissipation, or where the equations of motion can be written in the form \( \frac{dq_i}{dt} = \frac{p_i}{m_i} \), \( \frac{dp_i}{dt} = f(q_i) \).

**Note**

A short physical note: While the two-body-problem is known to be integrable, that means it can be solved with purely analytic techniques, already the three-body-problem is not solvable. This was found in the end of the 19th century by H. Poincare which led to the whole new subject of Chaos Theory.

**Define the system function**

To implement this system we define a 3D point type which will represent the space as well as the velocity. Therefore, we use the operators from Boost.Operators:
/*the point type */
template< class T , size_t Dim >
class point :
    boost::additive1< point< T , Dim > ,
    boost::additive2< point< T , Dim > , T ,
    boost::multiplicative2< point< T , Dim > , T > >
{
public:
    const static size_t dim = Dim;
    typedef T value_type;
    typedef point< value_type , dim > point_type;

    // ...
    // constructors

    // ...
    // operators

private:
    T m_val[dim];
};

The next step is to define a container type storing the values of q and p and to define system functions. As container type we use boost::array

// we simulate 5 planets and the sun
const size_t n = 6;

typedef point< double , 3 > point_type;
typedef boost::array< point_type , n > container_type;
typedef boost::array< double , n > mass_type;

The container_type is different from the state type of the ODE. The state type of the ode is simply a pair< container_type , container_type > since it needs the information about the coordinates and the momenta.

Next we define the system's equations. As we will use a stepper that accounts for the Hamiltonian (energy-preserving) character of the system, we have to define the rhs different from the usual case where it is just a single function. The stepper will make use of the separable character, which means the system will be defined by two objects representing \( f(p) = -\frac{dH}{dq} \) and \( g(q) = \frac{dH}{dp} \):

const double gravitational_constant = 2.95912208286e-4;

struct solar_system_coor
{
    const mass_type &m_masses;

    solar_system_coor( const mass_type &masses ) : m_masses( masses ) { }

    void operator() ( const container_type &p , container_type &dqdt ) const
    {
        for( size_t i=0 ; i<n ; ++i )
            dqdt[i] = p[i] / m_masses[i];
    }
};

15
In general a three body-system is chaotic, hence we can not expect that arbitrary initial conditions of the system will lead to a solution comparable with the solar system dynamics. That is we have to define proper initial conditions, which are taken from the book of Hairer, Wannier, Lubich [4].

As mentioned above, we need to use some special integrators in order to conserve phase space volume. There is a well known family of such integrators, the so-called Runge-Kutta-Nystroem solvers, which we apply here in terms of a symplectic_rkn_sb3a_mclachlan stepper:

```cpp
typedef symplectic_rkn_sb3a_mclachlan< container_type > stepper_type;
const double dt = 100.0;
integrate_const(
    stepper_type() ,
    make_pair( solar_system_coor( masses ) , solar_system_momentum( masses ) ) ,
    make_pair( boost::ref( q ) , boost::ref( p ) ) ,
    0.0 , 200000.0 , dt , streaming_observer( cout ) );
```

These integration routine was used to produce the above sketch of the solar system. Note, that there are two particularities in this example. First, the state of the symplectic stepper is not container_type but a pair of container_type. Hence, we must pass such a pair to the integrate function. Since, we want to pass them as references we can simply pack them into Boost.Ref. The second point is the observer, which is called with a state type, hence a pair of container_type. The reference wrapper is also passed, but this is not a problem at all:
**Chaotic systems and Lyapunov exponents**

In this example we present application of odeint to investigation of the properties of chaotic deterministic systems. In mathematical terms chaotic refers to an exponential growth of perturbations $\delta x$. In order to observe this exponential growth one usually solves the equations for the tangential dynamics which is again an ordinary differential equation. These equations are linear but time dependent and can be obtained via

$$\frac{d \delta x}{dt} = J(x) \delta x$$

where $J$ is the Jacobian of the system under consideration. $\delta x$ can also be interpreted as a perturbation of the original system. In principle $n$ of these perturbations exist, they form a hypercube and evolve in the time. The Lyapunov exponents are then defined as logarithmic growth rates of the perturbations. If one Lyapunov exponent is larger then zero the nearby trajectories diverge exponentially hence they are chaotic. If the largest Lyapunov exponent is zero one is usually faced with periodic motion. In the case of a largest Lyapunov exponent smaller then zero convergence to a fixed point is expected. More information's about Lyapunov exponents and nonlinear dynamical systems can be found in many textbooks, see for example: E. Ott "Chaos is Dynamical Systems", Cambridge.

To calculate the Lyapunov exponents numerically one usually solves the equations of motion for $n$ perturbations and orthonormalizes them every $k$ steps. The Lyapunov exponent is the average of the logarithm of the stretching factor of each perturbation.

To demonstrate how one can use odeint to determine the Lyapunov exponents we choose the Lorenz system. It is one of the most studied dynamical systems in the nonlinear dynamics community. For the standard parameters it possesses a strange attractor with non-integer dimension. The Lyapunov exponents take values of approximately 0.9, 0 and -12.

The implementation of the Lorenz system is

```cpp
struct streaming_observer
{
    std::ostream& m_out;

    streaming_observer( std::ostream &out ) : m_out( out ) { }

    template< class State >
    void operator()( const State &x , double t ) const
    {
        container_type &q = x.first;
        m_out << t;
        for( size_t i=0 ; i<q.size() ; ++i ) m_out << "\t" << q[i];
        m_out << "\n";
    }
};
```

**Tip**

You can use C++11 lambda to create the observers

The full example can be found here: [solar_system.cpp](https://example.com/solar_system.cpp)
const double sigma = 10.0;
const double R = 28.0;
const double b = 8.0 / 3.0;

typedef boost::array< double , 3 > lorenz_state_type;

void lorenz( const lorenz_state_type &x , lorenz_state_type &dxdt , double t )
{
    dxdt[0] = sigma * ( x[1] - x[0] );
    dxdt[1] = R * x[0] - x[1] - x[0] * x[2];
}

We need also to integrate the set of the perturbations. This is done in parallel to the original system, hence within one system function.
Of course, we want to use the above definition of the Lorenz system, hence the definition of the system function including the Lorenz system itself and the perturbation could look like:

code
const size_t n = 3;
const size_t num_of_lyap = 3;
const size_t N = n + n*num_of_lyap;
typedef std::tr1::array< double , N > state_type;
typedef std::tr1::array< double , num_of_lyap > lyap_type;

void lorenz_with_lyap( const state_type &x , state_type &dxdt , double t )
{
    lorenz( x , dxdt , t );
    for( size_t l=0 ; l<num_of_lyap ; ++l )
    {
        const double *pert = x.begin() + 3 + l * 3;
        double *dpert = dxdt.begin() + 3 + l * 3;
        dpert[0] = - sigma * pert[0] + 10.0 * pert[1];
    }
}

The perturbations are stored linearly in the state_type behind the state of the Lorenz system. The problem of lorenz() and lorenz_with_lyap() having different state types may be solved putting the Lorenz system inside a functor with templatized arguments:

code
struct lorenz
{
    template< class StateIn , class StateOut , class Value >
    void operator()( const StateIn &x , StateOut &dxdt , Value t )
    {
        dxdt[0] = sigma * ( x[1] - x[0] );
        dxdt[1] = R * x[0] - x[1] - x[0] * x[2];
    }
};

void lorenz_with_lyap( const state_type &x , state_type &dxdt , double t )
{
    lorenz()( x , dxdt , t );
    ...
}

This works fine and lorenz_with_lyap can be used for example via
This code snippet performs 1000 steps with constant step size 0.01.

A real world use case for the calculation of the Lyapunov exponents of Lorenz system would always include some transient steps, just to ensure that the current state lies on the attractor, hence it would look like

```cpp
state_type x;
// initialize x
explicit_rk4< state_type > rk4;
integrate_n_steps( rk4 , lorenz_with_lyap , x , 0.0 , 0.01 , 1000 );
```

The problem is now, that \( x \) is the full state containing also the perturbations and `integrate_n_steps` does not know that it should only use 3 elements. In detail, odeint and its steppers determine the length of the system under consideration by determining the length of the state. In the classical solvers, e.g. from Numerical Recipes, the problem was solved by pointer to the state and an appropriate length, something similar to

```cpp
void lorenz( double* x , double *dxdt , double t, void* params )
{
...
}
```

```cpp
int system_length = 3;
rk4( x , system_length , t , dt , lorenz );
```

But odeint supports a similar and much more sophisticated concept: `Boost.Range`. To make the steppers and the system ready to work with `Boost.Range` the system has to be changed:

```cpp
struct lorenz
{
    template< class State , class Deriv >
    void operator()( const State &x_ , Deriv &dxdt_ , double t ) const
    {
        typename boost::range_iterator< const State >::type x = boost::begin( x_ );
        typename boost::range_iterator< Deriv >::type dxdt = boost::begin( dxdt_ );
        dxdt[0] = sigma * ( x[1] - x[0] );
        dxdt[1] = R * x[0] - x[1] - x[0] * x[2];
    }
};
```

This is in principle all. Now, we only have to call `integrate_n_steps` with a range including only the first 3 components of \( x \):

```cpp
// perform 10000 transient steps
integrate_n_steps( rk4 , lorenz() , std::make_pair( x.begin() , x.begin() + n ) , 0.0 , dt , 10000 );
```

Having integrated a sufficient number of transients steps we are now able to calculate the Lyapunov exponents:

1. Initialize the perturbations. They are stored linearly behind the state of the Lorenz system. The perturbations are initialized such that \( p_{ij} = \delta_{ij} \), where \( p_{ij} \) is the \( j \)-component of the \( i \)-th perturbation and \( \delta_{ij} \) is the Kronecker symbol.

2. Integrate 100 steps of the full system with perturbations
3. Orthonormalize the perturbation using Gram-Schmidt orthonormalization algorithm.

4. Repeat step 2 and 3. Every 10000 steps write the current Lyapunov exponent.

```cpp
fill( x.begin()+n , x.end() , 0.0 );
for( size_t i=0 ; i<num_of_lyap ; ++i ) x[n+i] = 1.0;
fill( lyap.begin() , lyap.end() , 0.0 );

double t = 0.0;
size_t count = 0;
while( true )
{
    t = integrate_n_steps( rk4 , lorenz_with_lyap , x , t , dt , 100 );
    gram_schmidt< num_of_lyap >( x , lyap , n );
    ++count;
    if( !(count % 100000) )
    {
        cout << t;
        for( size_t i=0 ; i<num_of_lyap ; ++i ) cout << " \" << lyap[i] / t ;
        cout << endl;
    }
}
```

The full code can be found here: chaotic_system.cpp

**Stiff systems**

An important class of ordinary differential equations are so called stiff system which are characterized by two or more time scales of different order. Examples of such systems are found in chemical systems where reaction rates of individual sub-reaction might differ over large ranges, for example:

\[
\frac{dS_1}{dt} = -101 S_2 - 100 S_1 \\
\frac{dS_2}{dt} = S_1
\]

In order to efficiently solve stiff systems numerically the Jacobian

\[
J = \frac{df_i}{dx_j}
\]

is needed. Here is the definition of the above example
typedef boost::numeric::ublas::vector< double > vector_type;

typedef boost::numeric::ublas::matrix< double > matrix_type;

struct stiff_system
{
    void operator()( const vector_type &x , vector_type &dxdt , double /* t */ )
    {
        dxdt[0] = -101.0 * x[0] - 100.0 * x[1];
        dxdt[1] = x[0];
    }
};

struct stiff_system_jacobi
{
    void operator()( const vector_type & /* x */ , matrix_type &J , const double & /* t */ , vector_type &dfdt )
    {
        J(0,0) = -101.0;
        J(0,1) = -100.0;
        J(1,0) = 1.0;
        J(1,1) = 0.0;
        dfdt[0] = 0.0;
        dfdt[1] = 0.0;
    }
};

The state type has to be a ublas::vector and the matrix type must be a ublas::matrix since the stiff integrator only accepts these types. However, you might want use non-stiff integrators on this system, too - we will do so later for demonstration. Therefore we want to use the same function also with other state_types, realized by templatizing the operator():

typedef boost::numeric::ublas::vector< double > vector_type;

typedef boost::numeric::ublas::matrix< double > matrix_type;

struct stiff_system
{
    template< class State >
    void operator()( const State &x , State &dxdt , double t )
    {
        ...
    }
};

struct stiff_system_jacobi
{
    template< class State , class Matrix >
    void operator()( const State &x , Matrix &J , const double &t , State &dfdt )
    {
        ...
    }
};

Now you can use stiff_system in combination with std::vector or boost::array. In the example the explicit time derivative of f(x,t) is introduced separately in the Jacobian. If df/dt = 0 simply fill dfdt with zeros.

A well know solver for stiff systems is the Rosenbrock method. It has a step size control and dense output facilities and can be used like all the other steppers:
vector_type x( 2 , 1.0 );

size_t num_of_steps = integrate_const( make_dense_output< rosenbrock4< double > >( 1.0e-6 , 1.0e-6 ) ,
    make_pair( stiff_system() , stiff_system_jacobi() ) ,
    x , 0.0 , 50.0 , 0.01 ,
    cout << phoenix::arg_names::arg2 << " " << phoenix::arg_names::arg1[0] << "\n" );

During the integration 71 steps have been done. Comparing to a classical Runge-Kutta solver this is a very good result. For example
the Dormand-Prince 5 method with step size control and dense output yields 1531 steps.

vector_type x2( 2 , 1.0 );

size_t num_of_steps2 = integrate_const( make_dense_output< runge_kutta_dopri5< vector_type > >( 1.0e-6 , 1.0e-6 ) ,
    stiff_system() , x2 , 0.0 , 50.0 , 0.01 ,
    cout << phoenix::arg_names::arg2 << " " << phoenix::arg_names::arg1[0] << "\n" );

Note, that we have used Boost.Phoenix, a great functional programming library, to create and compose the observer.

The full example can be found here: stiff_system.cpp

**Complex state types**

Thus far we have seen several examples defined for real values. odeint can handle complex state types, hence ODEs which are
defined on complex vector spaces, as well. An example is the Stuart-Landau oscillator

\[
\frac{d}{dt} \Psi = ( 1 + i \eta ) \Psi + ( 1 + i \alpha ) | \Psi |^2 \Psi
\]

where \( \Psi \) and \( i \) is a complex variable. The definition of this ODE in C++ using complex< double > as a state type may look as follows

```cpp
typedef complex< double > state_type;

struct stuart_landau
{
    double m_eta;
    double m_alpha;

    stuart_landau( double eta = 1.0 , double alpha = 1.0 )
        : m_eta( eta ) , m_alpha( alpha ) { }

    void operator()( const state_type &x , state_type &dxdt , double t ) const
    {
        const complex< double > I( 0.0 , 1.0 );
        dxdt = ( 1.0 + m_eta * I ) * x - ( 1.0 + m_alpha * I ) * norm( x ) * x;
    }
};
```

One can also use a function instead of a functor to implement it

```cpp
double eta = 1.0;
double alpha = 1.0;

void stuart_landau( const state_type &x , state_type &dxdt , double t )
{
    const complex< double > I( 0.0 , 1.0 );
    dxdt[0] = ( 1.0 + m_eta * I ) * x[0] - ( 1.0 + m_alpha * I ) * norm( x[0] ) * x[0];
}
```
We strongly recommend to use the first ansatz. In this case you have explicit control over the parameters of the system and are not restricted to use global variables to parametrize the oscillator.

When choosing the stepper type one has to account for the "unusual" state type: it is a single `complex<double>` opposed to the vector types used in the previous examples. This means that no iterations over vector elements have to be performed inside the stepper algorithm. You can enforce this by supplying additional template arguments to the stepper including the vector_space_algebra. Details on the usage of algebras can be found in the section Adapt your own state types.

```cpp
state_type x = complex<double>(1.0, 0.0);
const double dt = 0.1;

typedef runge_kutta4<state_type,double,state_type,double,
vector_space_algebra> stepper_type;

integrate_const(stepper_type(), stuart_landau(2.0, 1.0), x, 0.0, 10.0, dt,
streaming_output(server(cout)));
```

The full cpp file for the Stuart-Landau example can be found here stuart_landau.cpp

**Note**
The fact that we have to configure a different algebra is solely due to the fact that we use a non-vector state type and not to the usage of complex values. So for, e.g. `vector<complex<double>>`, this would not be required.

**Lattice systems**

odeint can also be used to solve ordinary differential equations defined on lattices. A prominent example is the Fermi-Pasta-Ulam system [8]. It is a Hamiltonian system of nonlinear coupled harmonic oscillators. The Hamiltonian is

\[ H = \sum_i p_i^2/2 + 1/2 (q_{i+1} - q_i)^2 + \beta/4 (q_{i+1} - q_i)^4 \]

Remarkably, the Fermi-Pasta-Ulam system was the first numerical experiment to be implemented on a computer. It was studied at Los Alamos in 1953 on one of the first computers (a MANIAC I) and it triggered a whole new tree of mathematical and physical science.

Like the Solar System, the FPU is solved again by a symplectic solver, but in this case we can speed up the computation because the \( q \) components trivially reduce to \( dq_i / dt = p_i \). odeint is capable of doing this performance improvement. All you have to do is to call the symplectic solver with an state function for the \( p \) components. Here is how this function looks like
typedef vector< double > container_type;

struct fpu
{
  const double m_beta;

  fpu( const double beta = 1.0 ) : m_beta( beta ) { }

  // system function defining the ODE
  void operator()( const container_type &q , container_type &dpdt ) const
  {
    size_t n = q.size();
    double tmp = q[0] - 0.0;
    double tmp2 = tmp + m_beta * tmp * tmp * tmp;
    dpdt[0] = -tmp2;
    for( size_t i=0 ; i<n-1 ; ++i )
    {
       tmp = q[i+1] - q[i];
       tmp2 = tmp + m_beta * tmp * tmp * tmp;
       dpdt[i] += tmp2;
       dpdt[i+1] = -tmp2;
    }
    tmp = -q[n-1];
    tmp2 = tmp + m_beta * tmp * tmp * tmp;
    dpdt[n-1] += tmp2;
  }

  // calculates the energy of the system
  double energy( const container_type &q , const container_type &p ) const
  {
    // ...
  }

  // calculates the local energy of the system
  void local_energy( const container_type &q , const container_type &p , container_type &e ) const
  {
    // ...
  }
};

You can also use boost::array< double , N > for the state type.

Now, you have to define your initial values and perform the integration:
const size_t n = 64;
container_type q(n, 0.0), p(n, 0.0);

for(size_t i=0; i<n; ++i)
{
    p[i] = 0.0;
    q[i] = 32.0 * sin(double(i + 1) / double(n + 1) * M_PI);
}

const double dt = 0.1;
typedef symplectic_rkn_sb3a_mclachlan<container_type> stepper_type;
fpu fpu_instance(8.0);

integrate_const(stepper_type(), fpu_instance,
    make_pair(boost::ref(q), boost::ref(p)),
    0.0, 1000.0, dt, streaming_observer(cout, fpu_instance, 10));

The observer uses a reference to the system object to calculate the local energies:

```cpp
class streaming_observer
{
    std::ostream& m_out;
    const fpu &m_fpu;
    size_t m_write_every;
    size_t m_count;

    streaming_observer(std::ostream &out, const fpu &f, size_t write_every = 100)
        : m_out(out), m_fpu(f), m_write_every(write_every), m_count(0) {}

    template<class State>
    void operator()(const State &x, double t)
    {
        if((m_count % m_write_every) == 0)
        {
            container_type &q = x.first;
            container_type &p = x.second;
            container_type energy(q.size());
            m_fpu.local_energy(q, p, energy);
            for(size_t i=0; i<q.size(); ++i)
            {
                m_out << t << "\t" << i << "\t" << q[i] << "\t" << p[i] << "\t" << energy[i] << "\n";
                m_out << "\n";
                cerr << t << "\t" << accumulate(energy.begin(), energy.end(), 0.0) << "\n";
            }
            ++m_count;
        }
    }
};
```

The full cpp file for this FPU example can be found here fpu.cpp

**Ensembles of oscillators**

Another important high dimensional system of coupled ordinary differential equations is an ensemble of $N$ all-to-all coupled phase oscillators [9]. It is defined as

$$\frac{d\phi_k}{dt} = \omega_k + \varepsilon/N \sum_j \sin(\phi_j - \phi_k)$$
The natural frequencies $\omega_i$ of each oscillator follow some distribution and $\varepsilon$ is the coupling strength. We choose here a Lorentzian distribution for $\omega_i$. Interestingly a phase transition can be observed if the coupling strength exceeds a critical value. Above this value synchronization sets in and some of the oscillators oscillate with the same frequency despite their different natural frequencies. The transition is also called Kuramoto transition. Its behavior can be analyzed by employing the mean field of the phase

$$Z = K e^{i \Theta} = 1/N \sum_k e^{i \phi_k}$$

The definition of the system function is now a bit more complex since we also need to store the individual frequencies of each oscillator.

```cpp
typedef vector< double > container_type;

pair< double , double > calc_mean_field( const container_type &x )
{
    size_t n = x.size();
    double cos_sum = 0.0 , sin_sum = 0.0;
    for( size_t i=0 ; i<n ; ++i )
    {
        cos_sum += cos( x[i] );
        sin_sum += sin( x[i] );
    }
    cos_sum /= double( n );
    sin_sum /= double( n );
    double K = sqrt( cos_sum * cos_sum + sin_sum * sin_sum );
    double Theta = atan2( sin_sum , cos_sum );
    return make_pair( K , Theta );
}

struct phase_ensemble
{
    container_type m_omega;
    double m_epsilon;

    phase_ensemble( const size_t n , double g = 1.0 , double epsilon = 1.0 )
        : m_omega( n , 0.0 ) , m_epsilon( epsilon )
    {
        create_frequencies( g );
    }

    void create_frequencies( double g )
    {
        boost::mt19937 rng;
        boost::cauchy_distribution<> cauchy( 0.0 , g );
        boost::variate_generator< boost::mt19937&, boost::cauchy_distribution<> > gen( rng , cauchy );
        generate( m_omega.begin() , m_omega.end() , gen );
    }

    void set_epsilon( double epsilon ) { m_epsilon = epsilon; }
    double get_epsilon( void ) const { return m_epsilon; }

    void operator() ( const container_type &x , container_type &dxdt , double /* t */ ) const
```
Note, that we have used $Z$ to simplify the equations of motion. Next, we create an observer which computes the value of $Z$ and we record $Z$ for different values of $\epsilon$.

```cpp
struct statistics_observer
{
    double m_K_mean;
    size_t m_count;

    statistics_observer( void )
    : m_K_mean( 0.0 ), m_count( 0 ) { }

    template< class State >
    void operator()( const State &x, double t )
    {
        pair< double, double > mean = calc_mean_field( x );
        m_K_mean += mean.first;
        ++m_count;
    }

    double get_K_mean( void ) const { return ( m_count != 0 ) ? m_K_mean / double( m_count ) : 0.0; }

    void reset( void ) { m_K_mean = 0.0; m_count = 0; }
};
```

Now, we do several integrations for different values of $\epsilon$ and record $Z$. The result nicely confirms the analytical result of the phase transition, i.e. in our example the standard deviation of the Lorentzian is 1 such that the transition will be observed at $\epsilon = 2$. 

const size_t n = 16384;
const double dt = 0.1;

container_type x( n );

boost::mt19937 rng;
boost::uniform_real<> unif( 0.0 , 2.0 * M_PI );
boost::variate_generator< boost::mt19937&, boost::uniform_real<> > gen( rng , unif );

// gamma = 1, the phase transition occurs at epsilon = 2
phase_ensemble ensemble( n , 1.0 );
statistics_observer obs;

for( double epsilon = 0.0 ; epsilon < 5.0 ; epsilon += 0.1 )
{
    ensemble.set_epsilon( epsilon );
    obs.reset();
    // start with random initial conditions
    generate( x.begin() , x.end() , gen );
    // calculate some transients steps
    integrate_const( runge_kutta4< container_type >() , boost::ref( ensemble ) , x , 0.0 , 10.0 , dt );
    // integrate and compute the statistics
    integrate_const( runge_kutta4< container_type >() , boost::ref( ensemble ) , x , 0.0 , 100.0 , dt , boost::ref( obs ) );
    cout << epsilon << " \t " << obs.get_K_mean() << endl;
}

The full cpp file for this example can be found here phase_oscillator_ensemble.cpp

**Using boost::units**

odeint also works well with Boost.Units - a library for compile time unit and dimension analysis. It works by decoding unit information into the types of values. For a one-dimensional unit you can just use the Boost.Unit types as state type, deriv type and time type and hand the `vector_space_algebra` to the stepper definition and everything works just fine:

```cpp
typedef units::quantity< si::time , double > time_type;
typedef units::quantity< si::length , double > length_type;
typedef units::quantity< si::velocity , double > velocity_type;
typedef runge_kutta4< length_type , double , velocity_type , time_type ,
                    vector_space_algebra > stepper_type;
```

If you want to solve more-dimensional problems the individual entries typically have different units. That means that the `state_type` is now possibly heterogeneous, meaning that every entry might have a different type. To solve this problem, compile-time sequences from Boost.Fusion can be used.

To illustrate how odeint works with Boost.Units we use the harmonic oscillator as primary example. We start with defining all quantities
#include <boost/numeric/odeint.hpp>
#include <boost/numeric/odeint/algebra/fusion_algebra.hpp>

#include <boost/units/systems/si/length.hpp>
#include <boost/units/systems/si/time.hpp>
#include <boost/units/systems/si/velocity.hpp>
#include <boost/units/systems/si/acceleration.hpp>
#include <boost/units/systems/si/io.hpp>
#include <boost/fusion/container.hpp>

using namespace std;
using namespace boost::numeric::odeint;
namespace fusion = boost::fusion;
namespace units = boost::units;
namespace si = boost::units::si;

typedef units::quantity< si::time , double > time_type;
typedef units::quantity< si::length , double > length_type;
typedef units::quantity< si::velocity , double > velocity_type;
typedef units::quantity< si::acceleration , double > acceleration_type;
typedef units::quantity< si::frequency , double > frequency_type;
typedef fusion::vector< length_type , velocity_type > state_type;
typedef fusion::vector< velocity_type , acceleration_type > deriv_type;

Note, that the state_type and the deriv_type are now a compile-time fusion sequences. deriv_type represents \( x' \) and is now different from the state type as it has different unit definitions. Next, we define the ordinary differential equation which is completely equivalent to the example in Harmonic Oscillator:

struct oscillator
{
    frequency_type m_omega;

    oscillator( const frequency_type &omega = 1.0 * si::hertz ) : m_omega( omega ) ( )
    {
    }

    void operator()( const state_type &x , deriv_type &dxdt , time_type t ) const
    {
        fusion::at_c< 0 >( dxdt ) = fusion::at_c< 1 >( x );
        fusion::at_c< 1 >( dxdt ) = - m_omega * m_omega * fusion::at_c< 0 >( x );
    }
};

Next, we instantiate an appropriate stepper. We must explicitly parametrize the stepper with the state_type, deriv_type, time_type. Furthermore, the iteration over vector elements is now done by the fusion_algebra which must also be given. For more on the state types / algebras see chapter Adapt your own state types.

typedef runge_kutta_dopri5< state_type , double , deriv_type , time_type , fusion_algebra > step_per_type;

state_type x( 1.0 * si::meter , 0.0 * si::meter_per_second );

integrate_const( make_dense_output( 1.0e-6 , 1.0e-6 , stepper_type( ) ) , oscillator( 2.0 * si::hertz ) , x , 0.0 * si::second , 100.0 * si::second , 0.1 * si::second , streaming_observer( cout ) );

It is quite easy but the compilation time might take very long. Furthermore, the observer is defined a bit different
Caution

Using Boost.Units works nicely but compilation can be very time and memory consuming. For example the unit test for the usage of Boost.Units in odeint take up to 4 GB of memory at compilation.

The full cpp file for this example can be found here harmonic_oscillator_units.cpp.

Using matrices as state types

odeint works well with a variety of different state types. It is not restricted to pure vector-wise types, like vector<double>, array<double, N>, fusion::vector<double, double>, etc. but also works with types having a different topology then simple vectors. Here, we show how odeint can be used with matrices as states type, in the next section we will show how can be used to solve ODEs defined on complex networks.

By default, odeint can be used with ublas::matrix< T > as state type for matrices. A simple example is a two-dimensional lattice of coupled phase oscillators. Other matrix types like mtl::dense_matrix or blitz arrays and matrices can used as well but need some kind of activation in order to work with odeint. This activation is described in following sections.

The definition of the system is
typedef boost::numeric::ublas::matrix< double > state_type;

struct two_dimensional_phase_lattice
{
    two_dimensional_phase_lattice( double gamma = 0.5 )
    : m_gamma( gamma ) { }

    void operator()( const state_type &x , state_type &dxdt , double /* t */ ) const
    {
        size_t size1 = x.size1() , size2 = x.size2();
        for( size_t i=1 ; i<size1-1 ; ++i )
            for( size_t j=1 ; j<size2-1 ; ++j )
                dxdt( i , j ) =
                    coupling_func( x( i + 1 , j ) - x( i , j ) ) +
                    coupling_func( x( i - 1 , j ) - x( i , j ) ) +
                    coupling_func( x( i , j + 1 ) - x( i , j ) ) +
                    coupling_func( x( i , j - 1 ) - x( i , j ) );
        for( size_t i=0 ; i<x.size1() ; ++i ) dxdt( i , 0 ) = dxdt( i , x.size2() -1 ) = 0.0;
        for( size_t j=0 ; j<x.size2() ; ++j ) dxdt( 0 , j ) = dxdt( x.size1() -1 , j ) = 0.0;
    }

    double coupling_func( double x ) const
    {
        return sin( x ) - m_gamma * ( 1.0 - cos( x ) );
    }

    double m_gamma;
};

In principle this is all. Please note, that the above code is far from being optimal. Better performance can be achieved if every interaction is only calculated once and iterators for columns and rows are used. Below are some visualizations of the evolution of this lattice equation.
The full cpp for this example can be found here `two_dimensional_phase_lattice.cpp`.

**Using arbitrary precision floating point types**

Besides the classical floating point number like `float`, `double`, `complex< double >` you can also use arbitrary precision types, like the types from `gmp` and `mpfr`. But you have to be careful about instantiating any numbers.

For `gmp` types you have to set the default precision before any number is instantiated. This can be done by calling `mpf_set_default_prec( precision )` as the first function in your main program. Secondly, you can not use any global constant variables since they will not be set with the default precision you have already set.

Here is a simple example:
```cpp
typedef mpfr_class value_type;
typedef boost::array< value_type , 3 > state_type;

struct lorenz
{
    void operator()( const state_type &x , state_type &dxdt , value_type t ) const
    {
        const value_type sigma( 10.0 );
        const value_type R( 28.0 );
        const value_type b( value_type( 8.0 ) / value_type( 3.0 ) );

        dxdt[0] = sigma * ( x[1] - x[0] );
        dxdt[1] = R * x[0] - x[1] - x[0] * x[2];
    }
};
```

which can be integrated:

```cpp
const int precision = 1024;
mpf_set_default_prec( precision );

state_type x = {{ value_type( 10.0 ) , value_type( 10.0 ) , value_type( 10.0 ) }};

cout.precision( 1000 );
integrate_const( runge_kutta4< state_type , value_type >() ,
                lorenz() , x , value_type( 0.0 ) , value_type( 10.0 ) , value_type( value_type( 1.0 ) / value_type( 10.0 ) ) ,
                streaming_observer( cout ) );
```

**Caution**

The full support of arbitrary precision types depends on the functionality they provide. For example, the types from gmp are lacking of functions for calculating the power and arbitrary roots, hence they can not be used with the controlled steppers. In detail, for full support the \( \min(x,y) \), \( \max(x,y) \), \( \text{pow}(x,y) \) must be callable.

The full example can be found at lorenz_gmpxx.cpp.

**Self expanding lattices**

odeint supports changes of the state size during integration if a state_type is used which can be resized, like `std::vector`. The adjustment of the state's size has to be done from outside and the stepper has to be instantiated with `always_resizer` as the template argument for the `resizer_type`. In this configuration, the stepper checks for changes in the state size and adjust it's internal storage accordingly.

We show this for a Hamiltonian system of nonlinear, disordered oscillators with nonlinear nearest neighbor coupling.

The system function is implemented in terms of a class that also provides functions for calculating the energy. Note, that this class stores the random potential internally which is not resized, but rather a start index is kept which should be changed whenever the states' size change.
typedef vector< double > coord_type;
typedef pair< coord_type, coord_type > state_type;

struct compacton_lattice
{
    const int m_max_N;
    const double m_beta;
    int m_pot_start_index;
    vector< double > m_pot;

    compacton_lattice( int max_N , double beta , int pot_start_index )
    : m_max_N( max_N ) , m_beta( beta ) , m_pot_start_index( pot_start_index ) , m_pot( max_N )
    {
        srand( time( NULL ) );
        // fill random potential with iid values from [0,1]
        boost::mt19937 rng;
        boost::uniform_real<> unif( 0.0 , 1.0 );
        boost::variate_generator< boost::mt19937&, boost::uniform_real<> > gen( rng , unif );
        generate( m_pot.begin() , m_pot.end() , gen );
    }

    void operator()( const coord_type &q , coord_type &dpdt )
    {
        // calculate dpdt = -dH/dq of this hamiltonian system
        // dp_i/dt = - V_i * q_i^3 - beta*(q_i - q_{i-1})^3 + beta*(q_{i+1} - q_i)^3
        const int N = q.size();
        double diff = q[0] - q[N-1];
        for( int i=0 ; i<N ; ++i )
        {
            dpdt[i] = - m_pot[m_pot_start_index+i] * q[i]*q[i]*q[i] -
                      m_beta * diff*diff*diff;
            diff = q[(i+1) % N] - q[i];
            dpdt[i] += m_beta * diff*diff*diff;
        }
    }

    void energy_distribution( const coord_type &q , const coord_type &p , coord_type &energies )
    {
        // computes the energy per lattice site normalized by total energy
        const size_t N = q.size();
        double en = 0.0;
        for( size_t i=0 ; i<N ; i++ )
        {
            const double diff = q[(i+1) % N] - q[i];
            energies[i] = p[i]*p[i]/2.0
                          + m_pot[m_pot_start_index+i]*q[i]*q[i]*q[i]*q[i]/4.0
                          + m_beta/4.0 * diff*diff*diff*diff;
            en += energies[i];
        }
        en = 1.0/en;
        for( size_t i=0 ; i<N ; i++ )
        {
            energies[i] *= en;
        }
    }

    double energy( const coord_type &q , const coord_type &p )
    {
        // calculates the total energy of the excitation
        const size_t N = q.size();
        double en = 0.0;
        for( size_t i=0 ; i<N ; i++ )
        {
        }
const double diff = q[(i+1) % N] - q[i];
    en += p[i]*p[i]/2.0
    + m_pot[m_pot_start_index+i]*q[i]*q[i]*q[i]*q[i] / 4.0
    + m_beta/4.0 * diff*diff*diff*diff;
    return en;
}

void change_pot_start( const int delta )
{
    m_pot_start_index += delta;
}

The total size we allow is 1024 and we start with an initial state size of 60.

//start with 60 sites
const int N_start = 60;
coord_type q( N_start , 0.0 );
q.reserve( max_N );
coord_type p( N_start , 0.0 );
p.reserve( max_N );
// start with uniform momentum distribution over 20 sites
fill( p.begin()+20 , p.end()-20 , 1.0/sqrt(20.0) );
distr.reserve( max_N );
// create the system
compacton_lattice lattice( max_N , beta , (max_N-N_start)/2 );

//create the stepper, note that we use an always_resizer because state size might change during steps
typedef symplectic_rkn_sb3a_mclachlan< coord_type , coord_type , double , coord_type , coord_type ,
        range_algebra , default_operations , always_resizer > hamiltonian_stepper;
hamiltonian_stepper stepper;
hamiltonian_stepper::state_type state = make_pair( q , p );

The lattice gets resized whenever the energy distribution comes close to the borders distr[10] > 1E-150, distr[distr.size()-10] > 1E-150. If we increase to the left, q and p have to be rotated because their resize function always appends at the end. Additionally, the start index of the potential changes in this case.
double t = 0.0;
const double dt = 0.1;
const int steps = 10000;
for( int step = 0 ; step < steps ; ++step )
{
    stepper.do_step( boost::ref(lattice) , state , t , dt );
    lattice.energy_distribution( state.first , state.second , distr );
    if( distr[10] > 1E-150 )
    {
        do_resize( state.first , state.second , distr , state.first.size()+20 );
        rotate( state.first.begin() , state.first.end()-20 , state.first.end() );
        rotate( state.second.begin() , state.second.end()-20 , state.second.end() );
        lattice.change_pot_start( -20 );
        cout << t << " : resized left to " << distr.size() << ", energy = " << lattice.energy( state.first , state.second ) << endl;
    }
    if( distr[distr.size()-10] > 1E-150 )
    {
        do_resize( state.first , state.second , distr , state.first.size()+20 );
        cout << t << " : resized right to " << distr.size() << ", energy = " << lattice.energy( state.first , state.second ) << endl;
    }
    t += dt;
}

The do_resize function simply calls vector.resize of q, p and distr.

```cpp
void do_resize( coord_type &q , coord_type &p , coord_type &distr , const int N )
{
    q.resize( N );
    p.resize( N );
    distr.resize( N );
}
```

The full example can be found in resizing_lattice.cpp

## Using CUDA (or OpenMP, TBB, ...) via Thrust

Modern graphic cards (graphic processing units - GPUs) can be used to speed up the performance of time consuming algorithms by means of massive parallelization. They are designed to execute many operations in parallel. odeint can utilize the power of GPUs by means of CUDA and Thrust, which is a STL-like interface for the native CUDA API.

### Important

Thrust also supports parallelization using OpenMP and Intel Threading Building Blocks (TBB). You can switch between CUDA, OpenMP and TBB parallelizations by a simple compiler switch. Hence, this also provides an easy way to get basic OpenMP parallelization into odeint. The examples discussed below are focused on GPU parallelization, though.

To use odeint with CUDA a few points have to be taken into account. First of all, the problem has to be well chosen. It makes absolutely no sense to try to parallelize the code for a three dimensional system, it is simply too small and not worth the effort. One single function call (kernel execution) on the GPU is slow but you can do the operation on a huge set of data with only one call. We have experienced that the vector size over which is parallelized should be of the order of $10^6$ to make full use of the GPU. Secondly, you have to use Thrust’s algorithms and functors when implementing the rhs the ODE. This might be tricky since it involves some kind of functional programming knowledge.

Typical applications for CUDA and odeint are large systems, like lattices or discretizations of PDE, and parameter studies. We introduce now three examples which show how the power of GPUs can be used in combination with odeint.
Important

The full power of CUDA is only available for really large systems where the number of coupled ordinary differential equations is of order $N=10^6$ or larger. For smaller systems the CPU is usually much faster. You can also integrate an ensemble of different uncoupled ODEs in parallel as shown in the last example.

Phase oscillator ensemble

The first example is the phase oscillator ensemble from the previous section:

$$d\phi_k/dt = \omega_k + \epsilon/N \sum_j \sin(\phi_j - \phi_k).$$

It has a phase transition at $\epsilon = 2$ in the limit of infinite numbers of oscillators $N$. In the case of finite $N$ this transition is smeared out but still clearly visible.

Thrust and CUDA are perfectly suited for such kinds of problems where one needs a large number of particles (oscillators). We start by defining the state type which is a `thrust::device_vector`. The content of this vector lives on the GPU. If you are not familiar with this we recommend reading the *Getting started* section on the Thrust website.

```cpp
//change this to float if your device does not support double computation
typedef double value_type;

//change this to host_vector< ... > of you want to run on CPU
typedef thrust::device_vector< value_type > state_type;
```

Thrust follows a functional programming approach. If you want to perform a calculation on the GPU you usually have to call a global function like `thrust::for_each`, `thrust::reduce`, etc. with an appropriate local functor which performs the basic operation. An example is

```cpp
struct add_two
{
    template< class T >
    __host__ __device__
    void operator()( T &t ) const
    {
        t += T( 2 );
    }
};

// ...
thrust::for_each( x.begin() , x.end() , add_two() );
```

This code generically adds two to every element in the container $x$.

For the purpose of integrating the phase oscillator ensemble we need

- to calculate the system function, hence the r.h.s. of the ODE.
- this involves computing the mean field of the oscillator example, i.e. the values of $R$ and $\theta$

The mean field is calculated in a class `mean_field_calculator`
struct mean_field_calculator
{
    struct sin_functor : public thrust::unary_function< value_type , value_type >
    {
        __host__ __device__
        value_type operator()( value_type x) const
        {
            return sin( x );
        }
    };

    struct cos_functor : public thrust::unary_function< value_type , value_type >
    {
        __host__ __device__
        value_type operator()( value_type x) const
        {
            return cos( x );
        }
    };

    static std::pair< value_type , value_type > get_mean( const state_type &x )
    {
        value_type sin_sum = thrust::reduce(
            thrust::make_transform_iterator( x.begin() , sin_functor() ) ,
            thrust::make_transform_iterator( x.end() , sin_functor() ) );
        value_type cos_sum = thrust::reduce(
            thrust::make_transform_iterator( x.begin() , cos_functor() ) ,
            thrust::make_transform_iterator( x.end() , cos_functor() ) );
        cos_sum /= value_type( x.size() );
        sin_sum /= value_type( x.size() );
        value_type K = sqrt( cos_sum * cos_sum + sin_sum * sin_sum );
        value_type Theta = atan2( sin_sum , cos_sum );
        return std::make_pair( K , Theta );
    }
};

Inside this class two member structures sin_functor and cos_functor are defined. They compute the sine and the cosine of a value and they are used within a transform iterator to calculate the sum of sin(\( \phi_k \)) and cos(\( \phi_k \)). The classifiers __host__ and __device__ are CUDA specific and define a function or operator which can be executed on the GPU as well as on the CPU. The line

```cpp
value_type sin_sum = thrust::reduce(
    thrust::make_transform_iterator( x.begin() , sin_functor() ) ,
    thrust::make_transform_iterator( x.end() , sin_functor() ) );
```

performs the calculation of this sine-sum on the GPU (or on the CPU, depending on your thrust configuration).

The system function is defined via
class phase_oscillator_ensemble
{
  public:
    struct sys_functor
    {
      value_type m_K, m_Theta, m_epsilon;

      sys_functor(value_type K, value_type Theta, value_type epsilon)
      : m_K(K), m_Theta(Theta), m_epsilon(epsilon) {}

      template<class Tuple>
      __host__ __device__
      void operator()(Tuple t)
      {
        thrust::get<2>(t) = thrust::get<1>(t) + m_Theta * m_K * sin(m_Theta - thrust::get<0>(t));
      }
    }

    // ...

    void operator()(const state_type &x, state_type &dxdt, const value_type dt) const
    {
      std::pair<value_type, value_type> mean_field = mean_field_calculator::get_mean(x);

      thrust::for_each(
        thrust::make_zip_iterator(thrust::make_tuple(x.begin(), m_omega.begin(), dxdt.begin())),
        thrust::make_zip_iterator(thrust::make_tuple(x.end(), m_omega.end(), dxdt.end())),
        sys_functor(mean_field.first, mean_field.second, m_epsilon));
    }

    // ...
};

This class is used within the do_step and integrate method. It defines a member structure sys_functor for the r.h.s. of each individual oscillator and the operator() for the use in the steppers and integrators of odeint. The functor computes first the mean field of $\phi_k$ and secondly calculates the whole r.h.s. of the ODE using this mean field. Note, how nicely thrust::tuple and thrust::zip_iterator play together.

Now, we are ready to put everything together. All we have to do for making odeint ready for using the GPU is to parametrize the stepper with the appropriate thrust algebra/operations:

typedef runge_kutta4<state_type, value_type, state_type, value_type, thrust_algebra, thrust_operations> stepper_type;

You can also use a controlled or dense output stepper, e.g.

typedef runge_kutta_dopri5<state_type, value_type, state_type, value_type, thrust_algebra, thrust_operations> stepper_type;

Then, it is straightforward to integrate the phase ensemble by creating an instance of the rhs class and using an integration function:

phase_oscillator_ensemble ensemble(N, 1.0);
size_t steps1 = integrate_const( make_controlled( 1.0e-6 , 1.0e-6 , stepJper_type() ) , boost::ref( ensemble ) , x , 0.0 , t_transients , dt );

We have to use boost::ref here in order to pass the rhs class as reference and not by value. This ensures that the natural frequencies of each oscillator are not copied when calling integrate_const. In the full example the performance and results of the Runge-Kutta-4 and the Dopri5 solver are compared.

The full example can be found at phase_oscillator_example.cu.

**Large oscillator chains**

The next example is a large, one-dimensional chain of nearest-neighbor coupled phase oscillators with the following equations of motion:

\[
\frac{d \phi_k}{dt} = \omega_k + \sin( \phi_{k+1} - \phi_k ) + \sin( \phi_k - \phi_{k-1} )
\]

In principle we can use all the techniques from the previous phase oscillator ensemble example, but we have to take special care about the coupling of the oscillators. To efficiently implement the coupling you can use a very elegant way employing Thrust’s permutation iterator. A permutation iterator behaves like a normal iterator on a vector but it does not iterate along the usual order of the elements. It rather iterates along some permutation of the elements defined by some index map. To realize the nearest neighbor coupling we create one permutation iterator which travels one step behind a usual iterator and another permutation iterator which travels one step in front. The full system class is:
class phase_oscillators  

public:

struct sys_functor
{
  template< class Tuple >  
  __host__ __device__
  void operator()( Tuple t ) // this functor works on tuples of values
  {
    // first, unpack the tuple into value, neighbors and omega
    const value_type phi = thrust::get<0>(t);
    const value_type phi_left = thrust::get<1>(t); // left neighbor
    const value_type phi_right = thrust::get<2>(t); // right neighbor
    const value_type omega = thrust::get<3>(t);
    // the dynamical equation
    thrust::get<4>(t) = omega + sin( phi_right - phi ) + sin( phi - phi_left );
  }
};

phase_oscillators( const state_type &omega )
: m_omega( omega ) , m_N( omega.size() ) , m_prev( omega.size() ) , m_next( omega.size() )
{
  // build indices pointing to left and right neighbours
  thrust::counting_iterator<size_t> c( 0 );
  thrust::copy( c , c+m_N-1 , m_prev.begin()+1 );
  m_prev[0] = 0; // m_prev = { 0 , 0 , 1 , 2 , 3 , ... , N-1 }
  thrust::copy( c+1 , c+m_N , m_next.begin() );
  m_next[m_N-1] = m_N-1; // m_next = { 1 , 2 , 3 , ... , N-1 , N-1 }
}

void operator() ( const state_type &x , state_type &dxdt , const value_type dt )
{
  thrust::for_each(
    thrust::make_zip_iterator(
      thrust::make_tuple(x.begin(),
        thrust::make_permutation_iterator(x.begin(), m_prev.begin()),
        thrust::make_permutation_iterator(x.begin(), m_next.begin()),
        m_omega.begin(),
        dxdt.begin()))),
    thrust::make_zip_iterator(
      thrust::make_tuple(x.end(),
        thrust::make_permutation_iterator(x.begin(), m_prev.end()),
        thrust::make_permutation_iterator(x.begin(), m_next.end()),
        m_omega.end(),
        dxdt.end())) ) ,
  sys_functor();
}

private:
Note, how easy you can obtain the value for the left and right neighboring oscillator in the system functor using the permutation iterators. But, the call of the thrust::for_each function looks relatively complicated. Every term of the r.h.s. of the ODE is resembled by one iterator packed in exactly the same way as it is unpacked in the functor above.

Now we put everything together. We create random initial conditions and decreasing frequencies such that we should get synchronization. We copy the frequencies and the initial conditions onto the device and finally initialize and perform the integration. As result we simply write out the current state, hence the phase of each oscillator.

```cpp
// create initial conditions and omegas on host:
vector< value_type > x_host( N );
vector< value_type > omega_host( N );
for( size_t i=0 ; i<N ; ++i )
{
    x_host[i] = 2.0 * pi * drand48();
    omega_host[i] = ( N - i ) * epsilon; // decreasing frequencies
}

// copy to device
state_type x = x_host;
state_type omega = omega_host;

// create stepper
runge_kutta4< state_type , value_type , state_type , value_type , thrust_algebra , thrust_operations > stepper;

// create phase oscillator system function
phase_oscillators sys( omega );

// integrate
integrate_const( stepper , sys , x , 0.0 , 10.0 , dt );

thrust::copy( x.begin() , x.end() , std::ostream_iterator< value_type >( std::cout , "\n" ) );
std::cout << std::endl;
```

The full example can be found at phase_oscillator_chain.cu.

**Parameter studies**

Another important use case for Thrust and CUDA are parameter studies of relatively small systems. Consider for example the three-dimensional Lorenz system from the chaotic systems example in the previous section which has three parameters. If you want to study the behavior of this system for different parameters you usually have to integrate the system for many parameter values. Using thrust and odeint you can do this integration in parallel, hence you integrate a whole ensemble of Lorenz systems where each individual realization has a different parameter value.

In the following we will show how you can use Thrust to integrate the above mentioned ensemble of Lorenz systems. We will vary only the parameter \( \beta \) but it is straightforward to vary other parameters or even two or all three parameters. Furthermore, we will use the largest Lyapunov exponent to quantify the behavior of the system (chaoticity).

We start by defining the range of the parameters we want to study. The state_type is again a thrust::device_vector< value_type >.
The next thing we have to implement is the Lorenz system without perturbations. Later, a system with perturbations is also implemented in order to calculate the Lyapunov exponent. We will use an ansatz where each device function calculates one particular realization of the Lorenz ensemble.

```cpp
struct lorenz_system
{
    struct lorenz_functor
    {
        template< class T >
        __host__ __device__
        void operator()( T t ) const
        {
            // unpack the parameter we want to vary and the Lorenz variables
            value_type R = thrust::get< 3 >( t );
            value_type x = thrust::get< 0 >( t );
            value_type y = thrust::get< 1 >( t );
            value_type z = thrust::get< 2 >( t );
            thrust::get< 4 >( t ) = sigma * ( y - x );
            thrust::get< 5 >( t ) = R * x - y - x * z;
            thrust::get< 6 >( t ) = -b * z + x * y;
        }
    }

    lorenz_system( size_t N, const state_type &beta ) :
    m_N( N ), m_beta( beta ) {}

    template< class State, class Deriv >
    void operator()( const State &x, Deriv &dxdt, value_type t ) const
    {
        thrust::for_each(
            thrust::make_zip_iterator( thrust::make_tuple( boost::begin( x ),
                boost::begin( x ) + m_N,
                boost::begin( x ) + 2 * m_N,
                m_beta.begin(),
                boost::begin( dxdt ),
                boost::begin( dxdt ) + m_N,
                boost::begin( dxdt ) + 2 * m_N ) ),
            thrust::make_zip_iterator( thrust::make_tuple( boost::begin( x ) + m_N,
                boost::begin( x ) + 2 * m_N,
                boost::begin( x ) + 3 * m_N,
                m_beta.begin(),
                boost::begin( dxdt ) + m_N,
                boost::begin( dxdt ) + 2 * m_N,
                boost::begin( dxdt ) + 3 * m_N ) ),
            lorenz_functor() );
    }
}
```

size_t m_N;
const state_type &m_beta;
};
As state_type a thrust::device_vector or a Boost.Range of a device_vector is used. The length of the state is $3N$ where $N$ is the number of systems. The system is encoded into this vector such that all $x$ components come first, then every $y$ components and finally every $z$ components. Implementing the device function is then a simple task, you only have to decompose the tuple originating from the zip iterators.

Besides the system without perturbations we furthermore need to calculate the system including linearized equations governing the time evolution of small perturbations. Using the method from above this is straightforward, with a small difficulty that Thrust’s tuples have a maximal arity of 10. But this is only a small problem since we can create a zip iterator packed with zip iterators. So the top level zip iterator contains one zip iterator for the state, one normal iterator for the parameter, and one zip iterator for the derivative. Accessing the elements of this tuple in the system function is then straightforward, you unpack the tuple with thrust::get<>(). We will not show the code here, it is to large. It can be found here and is easy to understand.

Furthermore, we need an observer which determines the norm of the perturbations, normalizes them and averages the logarithm of the norm. The device functor which is used within this observer is defined

```cpp
struct lyap_functor
{
    template< class T >
    __host__ __device__
    void operator()( T t ) const
    {
        value_type &dx = thrust::get< 0 >( t );
        value_type &dy = thrust::get< 1 >( t );
        value_type &dz = thrust::get< 2 >( t );
        value_type norm = sqrt( dx * dx + dy * dy + dz * dz );
        dx /= norm;
        dy /= norm;
        dz /= norm;
        thrust::get< 3 >( t ) += log( norm );
    }
};
```

Note, that this functor manipulates the state, i.e. the perturbations.

Now we complete the whole code to calculate the Lyapunov exponents. First, we have to define a state vector. This vector contains $6N$ entries, the state $x,y,z$ and its perturbations $dx,dy,dz$. We initialize them such that $x=y=z=10$, $dx=1$, and $dy=dz=0$. We define a stepper type, a controlled Runge-Kutta Dormand-Prince 5 stepper. We start with some integration to overcome the transient behavior. For this, we do not involve the perturbation and run the algorithm only on the state $x,y,z$ without any observer. Note, how Boost.Range is used for partial integration of the state vector without perturbations (the first half of the whole state). After the transient, the full system with perturbations is integrated and the Lyapunov exponents are calculated and written to stdout.
state_type x(6 * N);

// initialize x,y,z
thrust::fill(x.begin(), x.begin() + 3 * N, 10.0);

// initial dx
thrust::fill(x.begin() + 3 * N, x.begin() + 4 * N, 1.0);

// initialize dy,dz
thrust::fill(x.begin() + 4 * N, x.end(), 0.0);

// create error stepper, can be used with make_controlled or make_dense_output
typedef runge_kutta_dopri5<state_type, value_type, state_type, value_type, thrust_algebra, thrust_operations> stepper_type;

lorenz_system lorenz(N, beta);
lorenz_perturbation_system lorenz_perturbation(N, beta);
lyap_observer obs(N, 1);

// calculate transients
integrate_adaptive(make_controlled(1.0e-6, 1.0e-6, step_per_type()), lorenz_perturbation, std::make_pair(x.begin(), x.begin() + 3 * N), 0.0, 10.0, dt);

// calculate the Lyapunov exponents -- the main loop
double t = 0.0;
while(t < 10000.0)
{
    integrate_adaptive(make_controlled(1.0e-6, 1.0e-6, stepper_type()), lorenz_perturbation, x, t, t + 1.0, 0.1);
    t += 1.0;
    obs(x, t);
}

vector<value_type> lyap(N);
obs.fill_lyap(lyap);

for(size_t i=0; i<N; ++i)
    cout << beta_host[i] << "\t" << lyap[i] << "\n";

The full example can be found at lorenz_parameters.cu.

**Using OpenCL via VexCL**

In the previous section the usage of odeint in combination with Thrust was shown. In this section we show how one can use OpenCL with odeint. The point of odeint is not to implement its own low-level data structures and algorithms, but to use high level libraries doing this task. Here, we will use the VexCL framework to use OpenCL. VexCL is a nice library for general computations and it uses heavily expression templates. With the help of VexCL it is possible to write very compact and expressive application.

**Note**

vexcl needs C++11 features! So you have to compile with C++11 support enabled.

To use VexCL one needs to include one additional header which includes the data-types and algorithms from vexcl and the adaption to odeint. Adaption to odeint means here only to adapt the resizing functionality of VexCL to odeint.

```
#include <boost/numeric/odeint/external/vexcl/vexcl_resize.hpp>
```
To demonstrate the use of VexCL, we integrate an ensemble of Lorenz systems. The example is very similar to the parameter study of the Lorenz system in the previous section except that we do not compute the Lyapunov exponents. Again, we vary the parameter R of the Lorenz system and solve a whole ensemble of Lorenz systems in parallel (each with a different parameter R). First, we define the state type and a vector type

```cpp
typedef vex::vector< double > vector_type;
typedef vex::multivector< double, 3 > state_type;
```

The `vector_type` is used to represent the parameter R. The `state_type` is a multi-vector of three sub vectors and is used to represent. The first component of this multi-vector represent all \( x \) components of the Lorenz system, while the second all \( y \) components and the third all \( z \) components. The components of this vector can be obtained via

```cpp
auto &x = X(0);
auto &y = X(1);
auto &z = X(2);
```

As already mentioned VexCL supports expression templates and we will use them to implement the system function for the Lorenz ensemble:

```cpp
const double sigma = 10.0;
const double b = 8.0 / 3.0;

struct sys_func :
{
    const vector_type &R;

    sys_func( const vector_type &_R ) : R( _R ) { }

    void operator()( const state_type &x , state_type &dxdt , double t ) const
    {
        dxdt(0) = -sigma * ( x(0) - x(1) );
        dxdt(1) = R * x(0) - x(1) - x(0) * x(2);
        dxdt(2) = - b * x(2) + x(0) * x(1);
    }
};
```

It’s very easy, isn’t it? These three little lines do all the computations for you. There is no need to write your own OpenCL kernels. VexCL does everything for you. Next we have to write the main application. We initialize the vector of parameters (R) and the initial state. Since VexCL supports odeint we can already use the `vector_space_algebra` in combination with the `default_operations` for the stepper and we are done:
// setup the opencl context
vex::Context ctx( vex::Filter::Type(CL_DEVICE_TYPE_GPU) );
std::cout << ctx << std::endl;

// set up number of system, time step and integration time
const size_t n = 1024 * 1024;
const double dt = 0.01;
const double t_max = 100.0;

// initialize R
double Rmin = 0.1, Rmax = 50.0, dR = (Rmax - Rmin) / double(n - 1);
std::vector<double> x(n * 3), r(n);
for(size_t i=0; i<n; ++i) r[i] = Rmin + dR * double(i);
vector_type R(ctx.queue(), r);

// initialize the state of the lorenz ensemble
state_type X(ctx.queue(), n);
X(0) = 10.0;
X(1) = 10.0;
X(2) = 10.0;

// create a stepper
runge_kutta4<
    state_type, double, state_type, double,
    odeint::vector_space_algebra, odeint::default_operations
> stepper;

// solve the system
integrate_const(stepper, sys_func(R), X, 0.0, t_max, dt);

---

**All examples**

The following table gives an overview over all examples.
Table 4. Examples Overview

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<tr>
<td>my_vector.cpp</td>
<td>Simple example showing how to get odeint to work with a self-defined vector type.</td>
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<tr>
<td>phase_oscillator_ensemble.cpp</td>
<td>The phase oscillator ensemble example shows how globally coupled oscillators can be analyzed and how statistical measures can be computed during integration.</td>
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<td>resizing_lattice.cpp</td>
<td>Shows the strength of odeint's memory management by simulating a Hamiltonian system on an expanding lattice.</td>
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<tr>
<td>-------------------------------------------</td>
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<tr>
<td>stochastic_euler.cpp</td>
<td>Implementation of a custom stepper - the stochastic euler - for solving stochastic differential equations.</td>
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<tr>
<td>stuart_landau.cpp</td>
<td>The Stuart-Landau example shows how odeint can be used with complex state types.</td>
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<tr>
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<td>The 2D phase oscillator example shows how a two-dimensional lattice works with odeint and how matrix types can be used as state types in odeint.</td>
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<tr>
<td>van_der_pol_stiff.cpp</td>
<td>This stiff system example again shows the usage of the stiff solvers by integrating the van der Pol oscillator.</td>
</tr>
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<td>gmpxx/lorenz_gmpxx.cpp</td>
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<tr>
<td>thrust/phase_oscillator_chain.cu</td>
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<td>thrust/relaxation.cu</td>
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<tr>
<td>ublas/lorenz_ublas.cpp</td>
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<td>vexcl/lorenz_ensemble.cpp</td>
<td>This example shows how the VexCL - a framework for OpenCL computation - can be used with odeint.</td>
</tr>
<tr>
<td>2d_lattice/spreading.cpp</td>
<td>This examples shows how a vector&lt; vector&lt; T &gt; &gt; can be used a state type for odeint and how a resizing mechanism of this state can be implemented.</td>
</tr>
<tr>
<td>quadmath/ black_hole.cpp</td>
<td>This examples shows how gcc libquadmath can be used with odeint. It provides a high precision floating point type which is adapted to odeint in this example.</td>
</tr>
</tbody>
</table>
odeint in detail

Steppers

Solving ordinary differential equation numerically is usually done iteratively, that is a given state of an ordinary differential equation is iterated forward \( x(t) \rightarrow x(t+dt) \rightarrow x(t+2dt) \). The steppers in odeint perform one single step. The most general stepper type is described by the Stepper concept. The stepper concepts of odeint are described in detail in section Concepts, here we briefly present the mathematical and numerical details of the steppers. The Stepper has two versions of the do_step method, one with an in-place transform of the current state and one with an out-of-place transform:

\[
\begin{align*}
\text{do_step} & ( \text{sys} , \text{inout} , t , dt ) \\
\text{do_step} & ( \text{sys} , \text{in} , t , \text{out} , dt )
\end{align*}
\]

The first parameter is always the system function - a function describing the ODE. In the first version the second parameter is the step which is here updated in-place and the third and the fourth parameters are the time and step size (the time step). After a call to do_step the state inout is updated and now represents an approximate solution of the ODE at time \( t+dt \). In the second version the second argument is the state of the ODE at time \( t \), the third argument is \( t \), the fourth argument is the approximate solution at time \( t+dt \) which is filled by do_step and the fifth argument is the time step. Note that these functions do not change the time \( t \).

System functions

Up to now, we have nothing said about the system function. This function depends on the stepper. For the explicit Runge-Kutta steppers this function can be a simple callable object hence a simple (global) C-function or a functor. The parameter syntax is \( \text{sys}(x, dxdt, t) \) and it is assumed that it calculates \( \frac{dx}{dt} = f(x,t) \). The function structure in most cases looks like:

```cpp
void sys( const state_type & /*x*/, state_type & /*dxdt*/, const double /*t*/ )
{
    // ...
}
```

Other types of system functions might represent Hamiltonian systems or systems which also compute the Jacobian needed in implicit steppers. For information which stepper uses which system function see the stepper table below. It might be possible that odeint will introduce new system types in near future. Since the system function is strongly related to the stepper type, such an introduction of a new stepper might result in a new type of system function.

Explicit steppers

A first specialization are the explicit steppers. Explicit means that the new state of the ode can be computed explicitly from the current state without solving implicit equations. Such steppers have in common that they evaluate the system at time \( t \) such that the result of \( f(x,t) \) can be passed to the stepper. In odeint, the explicit stepper have two additional methods

\[
\begin{align*}
\text{do_step} & ( \text{sys} , \text{inout} , dxdtin , t , dt ) \\
\text{do_step} & ( \text{sys} , \text{in} , dxdtin , t , \text{out} , dt )
\end{align*}
\]

Here, the additional parameter is the value of the function \( f \) at state \( x \) and time \( t \). An example is the Runge-Kutta stepper of fourth order:

```cpp
runge_kutta4< state_type > rk;
rk.do_step( sys1, inout, t, dt ); // In-place transformation of inout
rk.do_step( sys1, inout, t, dt ); // call with different system: Ok
rk.do_step( sys1, in, t, out, dt ); // Out-of-place transformation
rk.do_step( sys1, inout, dxdtin, t, dt ); // In-place transformation of inout
rk.do_step( sys1, in, dxdtin, t, out, dt ); // Out-of-place transformation
```
In fact, you do not need to call these two methods. You can always use the simpler `do_step(sys, inout, t, dt)`, but sometimes the derivative of the state is needed externally to do some external computations or to perform some statistical analysis.

A special class of the explicit steppers are the FSAL (first-same-as-last) steppers, where the last evaluation of the system function is also the first evaluation of the following step. For such steppers the `do_step` method are slightly different:

```cpp
    do_step(sys, inout, dxdtinout, t, dt)
    do_step(sys, in, dxdtin, out, dxdtout, t, dt)
```

This method takes the derivative at time $t$ and also stores the derivative at time $t+dt$. Calling these functions subsequently iterating along the solution one saves one function call by passing the result for $dxdt$ into the next function call. However, when using FSAL steppers without supplying derivatives:

```cpp
    do_step(sys, inout, t, dt)
```

the stepper internally satisfies the FSAL property which means it remembers the last $dxdt$ and uses it for the next step. An example for a FSAL stepper is the Runge-Kutta-Dopri5 stepper. The FSAL trick is sometimes also referred as the Fehlberg trick. An example how the FSAL steppers can be used is

```cpp
    runge_kutta_dopri5< state_type > rk;
    rk.do_step(sys1, in, t, out, dt);                       // DONT do this, sys1 is assumed
    rk.do_step(sys2, in, t, out, dt);                      // DONT do this, in2 is assumed
    rk.do_step(sys1, inout, dxdtinout, t, dt);            // Ok, internal derivative is not used, dxdtinout is updated
    rk.do_step(sys2, inout, dxdtinout, t, dt);            // Ok, internal derivative is not used
```

**Caution**

The FSAL-steppers save the derivative at time $t+dt$ internally if they are called via `do_step(sys, in, out, t, dt)`. The first call of `do_step` will initialize $dxdt$ and for all following calls it is assumed that the same system and the same state are used. If you use the FSAL stepper within the integrate functions this is taken care of automatically. See the Using steppers section for more details or look into the table below to see which stepper have an internal state.

**Symplectic solvers**

As mentioned above symplectic solvers are used for Hamiltonian systems. Symplectic solvers conserve the phase space volume exactly and if the Hamiltonian system is energy conservative they also conserve the energy approximately. A special class of symplectic systems are separable systems which can be written in the form $dq/dt = f_1(p)$, $dp/dt = f_2(q)$, where $(q,p)$ are the state of system.

The space of $(q,p)$ is sometimes referred as the phase space and $q$ and $p$ are said the be the phase space variables. Symplectic systems in this special form occur widely in nature. For example the complete classical mechanics as written down by Newton, Lagrange and Hamilton can be formulated in this framework. The separability of the system depends on the specific choice of coordinates.

Symplectic systems can be solved by odeint by means of the symplectic_euler stepper and a symplectic Runge-Kutta-Nystrom method of fourth order. These steppers assume that the system is autonomous, hence the time will not explicitly occur. Further they fulfill in principle the default Stepper concept, but they expect the system to be a pair of callable objects. The first entry of this pair calculates $f_1(p)$ while the second calculates $f_2(q)$. The syntax is `sys.first(p, dqdt)` and `sys.second(q, dpdt)`, where the first and second part can be again simple C-functions of functors. An example is the harmonic oscillator:
typedef boost::array< double , 1 > vector_type;

struct harm_osc_f1
{
    void operator()( const vector_type &p , vector_type &dqdt )
    {
        dqdt[0] = p[0];
    }
};

struct harm_osc_f2
{
    void operator()( const vector_type &q , vector_type &dpdt )
    {
        dpdt[0] = -q[0];
    }
};

The state of such an ODE consist now also of two parts, the part for q (also called the coordinates) and the part for p (the momenta). The full example for the harmonic oscillator is now:

pair< vector_type , vector_type > x;
x.first[0] = 1.0; x.second[0] = 0.0;
symplectic_rkn_sb3a_mclachlan< vector_type > rkn;
rkn.do_step( make_pair( harm_osc_f1() , harm_osc_f2() ) , x , t , dt );

If you like to represent the system with one class you can easily bind two public method:

struct harm_osc
{
    void f1( const vector_type &p , vector_type &dqdt ) const
    {
        dqdt[0] = p[0];
    }

    void f2( const vector_type &q , vector_type &dpdt ) const
    {
        dpdt[0] = -q[0];
    }
};

harm_osc h;
rkn.do_step( make_pair( boost::bind( &harm_osc::f1 , h , _1, _2 ) , boost::bind( &harm_osc::f2 , h , _1, _2 ) ) , x , t , dt );

Many Hamiltonian system can be written as \( dq/dt=p, \ dp/dt=f(q) \) which is computationally much easier than the full separable system. Very often, it is also possible to transform the original equations of motion to bring the system in this simplified form. This kind of system can be used in the symplectic solvers, by simply passing \( f(p) \) to the do_step method, again \( f(p) \) will be represented by a simple C-function or a functor. Here, the above example of the harmonic oscillator can be written as

pair< vector_type , vector_type > x;
x.first[0] = 1.0; x.second[0] = 0.0;
symplectic_rkn_sb3a_mclachlan< vector_type > rkn;
rkn.do_step( harm_osc_f1() , x , t , dt );

In this example the function harm_osc_f1 is exactly the same function as in the above examples.
Note, that the state of the ODE must not be constructed explicitly via `pair< vector_type , vector_type > x`. One can also use a combination of `make_pair` and `ref`. Furthermore, a convenience version of `do_step` exists which takes `q` and `p` without combining them into a pair:

```cpp
rkn.do_step( harm_osc_f1() , make_pair( boost::ref( q ) , boost::ref( p ) ) , t , dt );
rkn.do_step( harm_osc_f1() , q , p , t , dt );
rkn.do_step( make_pair( harm_osc_f1() , harm_osc_f2() ) , q , p , t , dt );
```

## Implicit solvers

### Caution

This section is not up-to-date.

For some kind of systems the stability properties of the classical Runge-Kutta are not sufficient, especially if the system is said to be stiff. A stiff system possesses two or more time scales of very different order. Solvers for stiff systems are usually implicit, meaning that they solve equations like \( x(t+dt) = x(t) + dt * f(x(t+1)) \). This particular scheme is the implicit Euler method. Implicit methods usually solve the system of equations by a root finding algorithm like the Newton method and therefore need to know the Jacobian of the system \( J_{ij} = \frac{df_i}{dx_j} \).

For implicit solvers the system is again a pair, where the first component computes \( f(x,t) \) and the second the Jacobian. The syntax is `sys.first( x , dxdt , t )` and `sys.second( x , J , t )`. For the implicit solver the `state_type` is `ublas::vector` and the Jacobian is represented by `ublas::matrix`.

### Important

Implicit solvers only work with `ublas::vector` as state type. At the moment, no other state types are supported.

## Multistep methods

Another large class of solvers are multi-step method. They save a small part of the history of the solution and compute the next step with the help of this history. Since multi-step methods know a part of their history they do not need to compute the system function very often, usually it is only computed once. This makes multi-step methods preferable if a call of the system function is expensive. Examples are ODEs defined on networks, where the computation of the interaction is usually where expensive (and might be of order \( O(N^2) \)).

Multi-step methods differ from the normal steppers. They save a part of their history and this part has to be explicitly calculated and initialized. In the following example an Adams-Bashforth-stepper with a history of 5 steps is instantiated and initialized;

```cpp
adams_bashforth_moulton< 5 , state_type > abm;
abm.initialize( sys , inout , t , dt );
abm.do_step( sys , inout , t , dt );
```

The initialization uses a fourth-order Runge-Kutta stepper and after the call of `initialize` the state of `inout` has changed to the current state, such that it can be immediately used by passing it to following calls of `do_step`. You can also use your own steppers to initialize the internal state of the Adams-Bashforth-Stepper:

```cpp
abm.initialize( runge_kutta_fehlberg78< state_type >() , sys , inout , t , dt );
```

Many multi-step methods are also explicit steppers, hence the parameter of `do_step` method do not differ from the explicit steppers.
Caution

The multi-step methods have some internal variables which depend on the explicit solution. Hence after any external changes of your state (e.g. size) or system the initialize function has to be called again to adjust the internal state of the stepper. If you use the integrate functions this will be taken into account. See the Using steppers section for more details.

Controlled steppers

Many of the above introduced steppers possess the possibility to use adaptive step-size control. Adaptive step size integration works in principle as follows:

1. The error of one step is calculated. This is usually done by performing two steps with different orders. The difference between these two steps is then used as a measure for the error. Stepper which can calculate the error are Error Stepper and they form an own class with an separate concept.

2. This error is compared against some predefined error tolerances. Are the tolerance violated the step is reject and the step-size is decreases. Otherwise the step is accepted and possibly the step-size is increased.

The class of controlled steppers has their own concept in odeint - the Controlled Stepper concept. They are usually constructed from the underlying error steppers. An example is the controller for the explicit Runge-Kutta steppers. The Runge-Kutta steppers enter the controller as a template argument. Additionally one can pass the Runge-Kutta stepper to the constructor, but this step is not necessary; the stepper is default-constructed if possible.

Different step size controlling mechanism exist. They all have in common that they somehow compare predefined error tolerance against the error and that they might reject or accept a step. If a step is rejected the step size is usually decreased and the step is made again with the reduced step size. This procedure is repeated until the step is accepted. This algorithm is implemented in the integration functions.

A classical way to decide whether a step is rejected or accepted is to calculate

\[
val = \frac{|| \text{err} ||}{\epsilon_{\text{abs}} + \epsilon_{\text{rel}} \times (a_x ||x|| + a_{dxdt} ||dxdt||)}
\]

\(\epsilon_{\text{abs}}\) and \(\epsilon_{\text{rel}}\) are the absolute and the relative error tolerances, and \(||x||\) is a norm, typically \(||x|| = (\sum_i x_i^2)^{1/2}\) or the maximum norm.

The step is rejected if \(val\) is greater then 1, otherwise it is accepted. For details of the used norms and error tolerance see the table below.

For the controlled_runge_kutta stepper the new step size is then calculated via

\[
val > 1 : dt_{\text{new}} = dt_{\text{current}} \times \max(0.9 \times val^{-1 / (O_E - 1)}, 0.2)
\]

\[
val < 0.5 : dt_{\text{new}} = dt_{\text{current}} \times \min(0.9 \times val^{-1 / O_S}, 5)
\]

\[
\text{else : } dt_{\text{new}} = dt_{\text{current}}
\]

Here, \(O_S\) and \(O_E\) are the order of the stepper and the error stepper. These formulas also contain some safety factors, avoiding that the step size is reduced or increased to much. For details of the implementations of the controlled steppers in odeint see the table below.
Table 5. Adaptive step size algorithms

<table>
<thead>
<tr>
<th>Stepper</th>
<th>Tolerance formula</th>
<th>Norm</th>
<th>Step size adaption</th>
</tr>
</thead>
<tbody>
<tr>
<td>controlled_runge_kutta</td>
<td>( \text{val} = \max( x_i ) )</td>
<td>(</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \text{else} : dt_{\text{new}} = dt_{\text{current}} )</td>
</tr>
<tr>
<td>rosenbrock4_controller</td>
<td>( \text{val} = \max( x_i ) )</td>
<td>( \text{val} = \max( x_i ) )</td>
<td>( \text{val} &gt; 1 : dt_{\text{new}} = dt_{\text{current}} \max(0.9 \ \text{pow}(\text{val}, -1 / (O_E - 1)), 0.2) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \text{else} : dt_{\text{new}} = dt_{\text{current}} )</td>
</tr>
<tr>
<td>bulirsch_stoer</td>
<td>( \text{tol} = 1/2 )</td>
<td>-</td>
<td>( dt_{\text{new}} = dt_{\text{old}} \ ^{1/a} )</td>
</tr>
</tbody>
</table>

To ease to generation of the controlled stepper, generation functions exist which take the absolute and relative error tolerances and a predefined error stepper and construct from this knowledge an appropriate controlled stepper. The generation functions are explained in detail in Generation functions.

**Dense output steppers**

A fourth class of stepper exists which are the so called dense output steppers. Dense-output steppers might take larger steps and interpolate the solution between two consecutive points. This interpolated points have usually the same order as the order of the stepper. Dense-output steppers are often composite stepper which take the underlying method as a template parameter. An example is the dense_output_runge_kutta stepper which takes a Runge-Kutta stepper with dense-output facilities as argument. Not all Runge-Kutta steppers provide dense-output calculation; at the moment only the Dormand-Prince 5 stepper provides dense output. An example is

```c++
dense_output_runge_kutta< controlled_runge_kutta< runge_kutta_dopri5< state_type > > > dense;
dense.initialize( in , t , dt );
pair< double , double > times = dense.do_step( sys );
```

Dense output stepper have their own concept. The main difference to usual steppers is that they manage the state and time internally. If you call do_step, only the ODE is passed as argument. Furthermore do_step return the last time interval: \( t \) and \( t + dt \), hence you can interpolate the solution between these two times points. Another difference is that they must be initialized with initialize, otherwise the internal state of the stepper is default constructed which might produce funny errors or bugs.

The construction of the dense output stepper looks a little bit nasty, since in the case of the dense_output_runge_kutta stepper a controlled stepper and an error stepper have to be nested. To simplify the generation of the dense output stepper generation functions exist:
typedef boost::numeric::odeint::result_of::make_dense_output<
  runge_kutta_dopri5< state_type > >::type dense_stepper_type;

dense_stepper_type dense2 = make_dense_output( 1.0e-6 , 1.0e-6
  , runge_kutta_dopri5< state_type >() );

This statement is also lengthy; it demonstrates how make_dense_output can be used with the result_of protocol. The parameters to make_dense_output are the absolute error tolerance, the relative error tolerance and the stepper. This explicitly assumes that the underlying stepper is a controlled stepper and that this stepper has an absolute and a relative error tolerance. For details about the generation functions see Generation functions. The generation functions have been designed for easy use with the integrate functions:

integrate_const( make_dense_output( 1.0e-6 , 1.0e-6
  , runge_kutta_dopri5< state_type >() ) , sys , inout , t_start , t_end , dt );

**Using steppers**

This section contains some general information about the usage of the steppers in odeint.

**Steppers are copied by value**

The stepper in odeint are always copied by values. They are copied for the creation of the controlled steppers or the dense output steppers as well as in the integrate functions.

**Steppers might have a internal state**

Some of the features described in this section are not yet implemented

Some steppers require to store some information about the state of the ODE between two steps. Examples are the multi-step methods which store a part of the solution during the evolution of the ODE, or the FSAL steppers which store the last derivative at time $t+dt$, to be used in the next step. In both cases the steppers expect that consecutive calls of do_step are from the same solution and the same ODE. In this case it is absolutely necessary that you call do_step with the same system function and the same state, see also the examples for the FSAL steppers above.

Stepper with an internal state support two additional methods: reset which resets the state and initialize which initializes the internal state. The parameters of initialize depend on the specific stepper. For example the Adams-Bashforth-Moulton stepper provides two initialize methods: initialize( system , inout , t , dt ) which initializes the internal states with the help of the Runge-Kutta 4 stepper, and initialize( stepper , system , inout , t , dt ) which initializes with the help of the stepper. For the case of the FSAL steppers, initialize is initialize( sys , in , t ) which simply calculates the r.h.s. of the ODE and assigns its value to the internal derivative.

All these steppers have in common, that they initially fill their internal state by themselves. Hence you are not required to call initialize. See how this works for the Adams-Bashforth-Moulton stepper: in the example we instantiate a fourth order Adams-Bashforth-Moulton stepper, meaning that it will store 4 internal derivatives of the solution at times $(t-dt, t-2*dt, t-3*dt, t-4*dt)$. 

Caution

Some of the features described in this section are not yet implemented
In the stepper table at the bottom of this page one can see which stepper have an internal state and hence provide the \texttt{reset} and \texttt{initialize} methods.

Stepper might be resizable

Nearly all steppers in odeint need to store some intermediate results of the type \texttt{state\_type} or \texttt{deriv\_type}. To do so odeint need some memory management for the internal temporaries. As this memory management is typically related to adjusting the size of vector-like types, it is called resizing in odeint. So, most steppers in odeint provide an additional template parameter which controls the size adjustment of the internal variables - the resizer. In detail odeint provides three policy classes (resizers) \texttt{always\_resizer}, \texttt{initially\_resizer}, and \texttt{never\_resizer}. Furthermore, all stepper have a method \texttt{adjust\_size} which takes a parameter representing a state type and which manually adjusts the size of the internal variables matching the size of the given instance. Before performing the actual resizing odeint always checks if the sizes of the state and the internal variable differ and only resizes if they are different.

\begin{itemize}
\item \texttt{Initially\_resizer} - the first call to \texttt{do\_step} performs the resizing, hence memory allocation. If you have changed the size of your system and your state you have to call \texttt{adjust\_size} by hand in this case.
\item \texttt{Always\_resizer} - tries to resize the internal variables at every call of \texttt{do\_step}. Typical use cases for this kind of resizer are self expanding lattices like shown in the tutorial (Self expanding lattices) or partial differential equations with an adaptive grid. Here, no calls of \texttt{adjust\_size} are required, the steppers manage everything themselves.
\item \texttt{Never\_resizer} - the internal variables are never adjusted automatically and always have to be adjusted by hand.
\end{itemize}

There is a second mechanism which influences the resizing and which controls if a state type is at least resizeable - a meta-function \texttt{is\_resizeable}. This meta-function returns a static Boolean value if any type is resizeable. For example it will return \texttt{true} for \texttt{std::vector<T>} but \texttt{false} for \texttt{boost::array<T>}. By default and for unknown types \texttt{is\_resizeable} returns \texttt{false}, so if you have your own type you need to specialize this meta-function. For more details on the resizing mechanism see the section Adapt your own state types.

Which steppers should be used in which situation

odeint provides a quite large number of different steppers such that the user is left with the question of which stepper fits his needs. Our personal recommendations are:

\begin{verbatim}
adams_bashforth_moulton< 4 , state\_type > stepper;
stepper.do\_step( sys , x , t , dt ); // make one step with the classical Runge-Kutta stepper and initialize the first internal state
   // the internal array is now [x(t-dt)]
stepper.do\_step( sys , x , t , dt ); // make one step with the classical Runge-Kutta stepper and initialize the second internal state
   // the internal state array is now [x(t-dt), x(t-2*dt)]
stepper.do\_step( sys , x , t , dt ); // make one step with the classical Runge-Kutta stepper and initialize the third internal state
   // the internal state array is now [x(t-dt), x(t-2*dt), x(t-3*dt)]
stepper.do\_step( sys , x , t , dt ); // make one step with Adam-Bashforth-Moulton, the internal array of states is now rotated
\end{verbatim}
• **runge_kutta_dopri5** is maybe the best default stepper. It has step size control as well as dense-output functionality. Simple create a dense-output stepper by `make_dense_output(1.0e-6, 1.0e-5, runge_kutta_dopri5<state_type>())`.

• **runge_kutta4** is a good stepper for constant step sizes. It is widely used and very well known. If you need to create artificial time series this stepper should be the first choice.

• 'runge_kutta_fehlberg78' is similar to the 'runge_kutta4' with the advantage that it has higher precision. It can also be used with step size control.

• **adams_bashforth_moulton** is very well suited for ODEs where the r.h.s. is expensive (in terms of computation time). It will calculate the system function only once during each step.
Stepper overview
## Table 6. Stepper Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Class</th>
<th>Concept</th>
<th>System Concept</th>
<th>Order</th>
<th>Error Estimation</th>
<th>Dense Output</th>
<th>Internal state</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit Euler</td>
<td>euler</td>
<td>Dense Output</td>
<td>System</td>
<td>1</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Very simple, only for demonstrating purpose</td>
</tr>
<tr>
<td>Modified Midpoint</td>
<td>modified_midpoint</td>
<td>Stepper</td>
<td>System</td>
<td>configurable (2)</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Used in Bulirsch-Stoer implementation</td>
</tr>
<tr>
<td>Runge-Kutta 4</td>
<td>runge_kutta4</td>
<td>Stepper</td>
<td>System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>The classical Runge-Kutta scheme, good general scheme without error control</td>
</tr>
<tr>
<td>Cash-Karp</td>
<td>runge_kutta5</td>
<td>Error Stepper</td>
<td>System</td>
<td>5</td>
<td>Yes (4)</td>
<td>No</td>
<td>No</td>
<td>Good general scheme with error estimation, to be used in controlled_error_stepper</td>
</tr>
<tr>
<td>Dormand-Prince 5</td>
<td>runge_kutta5</td>
<td>Error Stepper</td>
<td>System</td>
<td>5</td>
<td>Yes (4)</td>
<td>Yes</td>
<td>Yes</td>
<td>Standard method with error control and dense output, to be used in controlled_error_stepper and in dense_output_controlled_explicit_fsal.</td>
</tr>
<tr>
<td>Fehlberg 78</td>
<td>runge_kutta5</td>
<td>Error Stepper</td>
<td>System</td>
<td>8</td>
<td>Yes (7)</td>
<td>No</td>
<td>No</td>
<td>Good high order method with error estimation, to be used in controlled_error_stepper.</td>
</tr>
<tr>
<td>Algorithm</td>
<td>Class</td>
<td>Concept</td>
<td>System Concept</td>
<td>Order</td>
<td>Error Estimation</td>
<td>Dense Output</td>
<td>Internal State</td>
<td>Remarks</td>
</tr>
<tr>
<td>----------------------------</td>
<td>----------------------</td>
<td>--------------------</td>
<td>----------------</td>
<td>------------------------</td>
<td>------------------</td>
<td>--------------</td>
<td>----------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>Adams Bashforth</td>
<td>adams_bashforth</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Multistep method</td>
</tr>
<tr>
<td>Adams Moulton</td>
<td>adams_moulton</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Multistep method</td>
</tr>
<tr>
<td>Adams Bashforth Moulton</td>
<td>adams_bashforth_moulton</td>
<td>Stepper</td>
<td>System</td>
<td>configurable</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Combined multistep method</td>
</tr>
<tr>
<td>Controlled Runge-Kutta</td>
<td>controlled_runge_kutta</td>
<td>Controlled Stepper</td>
<td>System</td>
<td>depends</td>
<td>Yes</td>
<td>No</td>
<td>depends</td>
<td>Error control for Error Stepper. Requires an Error Stepper from above. Order depends on the given Error Stepper.</td>
</tr>
<tr>
<td>Dense Output Runge-Kutta</td>
<td>dense_output_runge_kutta</td>
<td>Dense Output Stepper</td>
<td>System</td>
<td>depends</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Dense output for Stepper and Error Stepper from above if they provide dense output functionality (like euler and runge_kutta_dopri5). Order depends on the given stepper.</td>
</tr>
<tr>
<td>Bulirsch-Stoer</td>
<td>bulirsch_stoer</td>
<td>Controlled Stepper</td>
<td>System</td>
<td>variable</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Stepper with step size and order control. Very good if high precision is required.</td>
</tr>
</tbody>
</table>

Remarks

Dense Output

Error Control for Error Stepper. Requires an Error Stepper from above. Order depends on the given Error Stepper.
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Class</th>
<th>Concept</th>
<th>System Concept</th>
<th>Order</th>
<th>Error Estimation</th>
<th>Dense Output</th>
<th>Internal state</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulirsch-Stoer Dense Output</td>
<td>bulbirsch_stoer_dense_out</td>
<td>Dense Output Stepper</td>
<td>System</td>
<td>variable</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Stepper with step size and order control as well as dense output. Very good if high precision and dense output is required.</td>
</tr>
<tr>
<td>Implicit Euler</td>
<td>implicit_euler</td>
<td>Stepper</td>
<td>Implicit System</td>
<td>1</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Basic implicit routine. Requires the Jacobian. Works only with BoostuBLAS vectors as state types.</td>
</tr>
<tr>
<td>Rosenbrock 4</td>
<td>rosenbrock4</td>
<td>Error Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Good for stiff systems. Works only with BoostuBLAS vectors as state types.</td>
</tr>
<tr>
<td>Controlled Rosenbrock 4</td>
<td>rosenbrock4_controller</td>
<td>Controlled Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Rosenbrock 4 with error control. Works only with BoostuBLAS vectors as state types.</td>
</tr>
<tr>
<td>Dense Output Rosenbrock 4</td>
<td>rosenbrock4_dense_output</td>
<td>Dense Output Stepper</td>
<td>Implicit System</td>
<td>4</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Controlled Rosenbrock 4 with dense output. Works only with BoostuBLAS vectors as state types.</td>
</tr>
</tbody>
</table>
**Custom steppers**

Finally, one can also write new steppers which are fully compatible with odeint. They only have to fulfill one or several of the stepper Concepts of odeint.

We will illustrate how to write your own stepper with the example of the stochastic Euler method. This method is suited to solve stochastic differential equations (SDEs). A SDE has the form

\[
dx/dt = f(x) + g(x) \xi(t)
\]

where \( \xi \) is Gaussian white noise with zero mean and a standard deviation \( \sigma(t) \). \( f(x) \) is said to be the deterministic part while \( g(x) \xi \) is the noisy part. In case \( g(x) \) is independent of \( x \) the SDE is said to have additive noise. It is not possible to solve SDE with the classical solvers for ODEs since the noisy part of the SDE has to be scaled differently then the deterministic part with respect to the time step. But there exist many solvers for SDEs. A classical and easy method is the stochastic Euler solver. It works by iterating

\[
x(t+\Delta t) = x(t) + \Delta t f(x(t)) + \Delta t^{1/2} g(x) \xi(t)
\]

where \( \xi(t) \) is an independent normal distributed random variable.

Now we will implement this method. We will call the stepper stochastic_euler. It models the Stepper concept. For simplicity, we fix the state type to be an `array< double , N >` The class definition looks like

---

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Class</th>
<th>Concept</th>
<th>System Concept</th>
<th>Order</th>
<th>Error Estimation</th>
<th>Dense Output</th>
<th>Internal State</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symplectic Euler</td>
<td>symplectic_euler</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>1</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Basic symplectic solver for separable Hamiltonian system</td>
</tr>
<tr>
<td>Symplectic R K N McLachlan</td>
<td>symplectic_euler_mclachlan</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Symplectic solver for separable Hamiltonian system with 6 stages and order 4.</td>
</tr>
<tr>
<td>Symplectic R K N McLachlan</td>
<td>symplectic_euler_mclachlan</td>
<td>Stepper</td>
<td>Symplectic System</td>
<td>4</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Symplectic solver with 5 stages and order 4, can be used with arbitrary precision types.</td>
</tr>
</tbody>
</table>
The types are needed in order to fulfill the stepper concept. As internal state and deriv type we use simple arrays in the stochastic Euler, they are needed for the temporaries. The stepper has the order one which is returned from the `order()` function.

The system functions need to calculate the deterministic and the stochastic part of our stochastic differential equation. So it might be suitable that the system function is a pair of functions. The first element of the pair computes the deterministic part and the second the stochastic one. Then, the second part also needs to calculate the random numbers in order to simulate the stochastic process. We can now implement the `do_step` method.

```cpp
template< size_t N > class stochastic_euler
{
    public:

        // ...

        template< class System >
        void do_step( System system , state_type &x , time_type t , time_type dt ) const
        {
            deriv_type det , stoch ;
            system.first( x , det );
            system.second( x , stoch );
            for( size_t i=0 ; i<x.size() ; ++i )
                x[i] += dt * det[i] + sqrt( dt ) * stoch[i];
        }
};
```

This is all. It is quite simple and the stochastic Euler stepper implement here is quite general. Of course it can be enhanced, for example

- use of operations and algebras as well as the resizing mechanism for maximal flexibility and portability
- use of `boost::ref` for the system functions
- use of `boost::range` for the state type in the `do_step` method
- ...

Now, let's look how we use the new stepper. A nice example is the Ornstein-Uhlenbeck process. It consists of a simple Brownian motion overlapped with an relaxation process. Its SDE reads

\[
dx/dt = -x + \xi
\]

where \( \xi \) is Gaussian white noise with standard deviation \( \sigma \). Implementing the Ornstein-Uhlenbeck process is quite simple. We need two functions or functors - one for the deterministic and one for the stochastic part:
const static size_t N = 1;
typedef boost::array< double , N > state_type;

struct ornstein_det
{
    void operator()( const state_type &x , state_type &dxdt ) const
    {
        dxdt[0] = -x[0];
    }
};

struct ornstein_stoch
{
    boost::mt19937 m_rng;
    boost::normal_distribution<> m_dist;

    ornstein_stoch( double sigma ) : m_rng() , m_dist( 0.0 , sigma ) { }

    void operator()( const state_type &x , state_type &dxdt )
    {
        dxdt[0] = m_dist( m_rng );
    }
};

In the stochastic part we have used the Mersenne twister for the random number generation and a Gaussian white noise generator normal_distribution with standard deviation \( \sigma \). Now, we can use the stochastic Euler stepper with the integrate functions:

```c++
double dt = 0.1;
state_type x = {{ 1.0 }};
integrate_const( stochastic_euler< N >() , make_pair( ornstein_det() , ornstein_stoch( 1.0 ) ) ,
    x , 0.0 , 10.0 , dt , streaming_observer() );
```

Note, how we have used the make_pair function for the generation of the system function.

## Custom Runge-Kutta steppers

odeint provides a C++ template meta-algorithm for constructing arbitrary Runge-Kutta schemes\(^1\). Some schemes are predefined in odeint, for example the classical Runge-Kutta of fourth order, or the Runge-Kutta-Cash-Karp 54 and the Runge-Kutta-Fehlberg 78 method. You can use this meta algorithm to construct you own solvers. This has the advantage that you can make full use of odeint’s algebra and operation system.

Consider for example the method of Heun, defined by the following Butcher tableau:

\[
\begin{array}{ccc}
  c_1 &=& 0 \\
  c_2 &=& 1/3, \quad a_{21} = 1/3 \\
  c_3 &=& 2/3, \quad a_{31} = 0, \quad a_{32} = 2/3 \\
\end{array}
\]

\[
\begin{array}{ccc}
  b_1 &=& 1/4, \quad b_2 = 0, \quad b_3 = 3/4 \\
\end{array}
\]

Implementing this method is very easy. First you have to define the constants:

\(^1\) M. Mulansky, K. Ahnert, Template-Metaprogramming applied to numerical problems, arxiv:1110.3233
While this might look cumbersome, packing all parameters into a templated class which is not immediately evaluated has the advantage that you can change the value type of your stepper to any type you like - presumably arbitrary precision types. One could also instantiate the coefficients directly:

```cpp
class HeunA1 {
public:
  HeunA1() {
    (*this)[0] = 1.0 / 3.0;
  }
private:
  static_cast<double>(1.0) / static_cast<double>(3.0);
};
```

```cpp
class HeunA2 {
public:
  HeunA2() {
    (*this)[0] = 0.0;
    (*this)[1] = 2.0 / 3.0;
  }
private:
  static_cast<double>(0.0) / static_cast<double>(3.0);
};
```

```cpp
class HeunB {
public:
  HeunB() {
    (*this)[0] = 1.0 / 4.0;
    (*this)[1] = 0.0;
    (*this)[2] = 3.0 / 4.0;
  }
private:
  static_cast<double>(1.0) / static_cast<double>(4.0); static_cast<double>(0.0); static_cast<double>(3.0) / static_cast<double>(4.0);
};
```

```cpp
class HeunC {
public:
  HeunC() {
    (*this)[0] = 0.0;
    (*this)[1] = 1.0 / 3.0;
    (*this)[2] = 2.0 / 3.0;
  }
private:
  static_cast<double>(0.0); static_cast<double>(1.0) / static_cast<double>(3.0); static_cast<double>(2.0) / static_cast<double>(3.0);
};
```

But then you are nailed down to use doubles.

Next, you need to define your stepper, note that the Heun method has 3 stages and produces approximations of order 3:
template<
    class State ,
    class Value = double ,
    class Deriv = State ,
    class Time = Value ,
    class Algebra = boost::numeric::odeint::range_algebra ,
    class Operations = boost::numeric::odeint::default_operations ,
    class Resizer = boost::numeric::odeint::initially_resizer
>
class heun : public
    boost::numeric::odeint::explicit_generic_rk< 3 , 3 , State , Value , Deriv , Time ,
    Algebra , Operations , Resizer >
{
public:
    typedef boost::numeric::odeint::explicit_generic_rk< 3 , 3 , State , Value , Deriv , Time ,
    Algebra , Operations , Resizer > step
    per_base_type;
    typedef typename stepper_base_type::state_type state_type;
    typedef typename stepper_base_type::wrapped_state_type wrapped_state_type;
    typedef typename stepper_base_type::value_type value_type;
    typedef typename stepper_base_type::deriv_type deriv_type;
    typedef typename stepper_base_type::wrapped_deriv_type wrapped_deriv_type;
    typedef typename stepper_base_type::time_type time_type;
    typedef typename stepper_base_type::algebra_type algebra_type;
    typedef typename stepper_base_type::operations_type operations_type;
    typedef typename stepper_base_type::resizer_type resizer_type;
    typedef typename stepper_base_type::stepper_type stepper_type;

    heun( const algebra_type &algebra = algebra_type() )
    : stepper_base_type(
        fusion::make_vector(
            heun_a1<Value>() ,
            heun_a2<Value>() ) ,
        heun_b<Value>() , heun_c<Value>() , algebra )
    {};

That’s it. Now, we have a new stepper method and we can use it, for example with the Lorenz system:

typedef boost::array< double , 3 > state_type;
heun< state_type > h;
state_type x = {{ 10.0 , 10.0 , 10.0 }};
integrate_const( h , lorenz() , x , 0.0 , 100.0 , 0.01 ,
    streaming_observer( std::cout ) );

Generation functions

In the Tutorial we have learned how we can use the generation functions make_controlled and make_dense_output to create controlled and dense output stepper from a simple stepper or an error stepper. The syntax of these two functions is very simple:

auto stepper1 = make_controlled( 1.0e-6 , 1.0e-6 , stepper_type() );
auto stepper2 = make_dense_output( 1.0e-6 , 1.0e-6 , stepper_type() );

The first two parameters are the absolute and the relative error tolerances and the third parameter is the stepper. In C++03 you can infer the type from the result_of mechanism:
To use your own steppers with the `make_controlled` or `make_dense_output` you need to specialize two class templates. Suppose your steppers are called `custom_stepper`, `custom_controller` and `custom_dense_output`. Then, the first class you need to specialize is `boost::numeric::odeint::get_controller`, a meta function returning the type of the controller:

```cpp
namespace boost { namespace numeric { namespace odeint {

    template<>
    struct get_controller< custom_stepper >
    {
        typedef custom_controller type;
    };

}}}
```

The second one is a factory class `boost::numeric::odeint::controller_factory` which constructs the controller from the tolerances and the stepper. In our dummy implementation this class is:

```cpp
namespace boost { namespace numeric { namespace odeint {

    template<>
    struct controller_factory< custom_stepper , custom_controller >
    {
        custom_controller operator()( double abs_tol , double rel_tol , const custom_stepper & ) const
        {
            return custom_controller();
        };
    };

}}}
```

This is all to use the `make_controlled` mechanism. Now you can use your controller via

```cpp
auto stepper5 = make_controlled( 1.0e-6 , 1.0e-6 , custom_stepper() );
```

For the `dense_output_stepper` everything works similar. Here you have to specialize `boost::numeric::odeint::get_dense_output` and `boost::numeric::odeint::dense_output_factory`. These two classes have the same syntax as their relatives `get_controller` and `controller_factory`.

All controllers and dense-output steppers in odeint can be used with these mechanisms. In the table below you will find, which steppers is constructed from `make_controlled` or `make_dense_output` if applied on a stepper from odeint:

```cpp
boost::numeric::odeint::result_of::make_controlled< stepper_type >::type stepper3 = make_controlled( 1.0e-6 , 1.0e-6 , stepper_type() );
boost::numeric::odeint::result_of::make_dense_output< stepper_type >::type step4 = make_dense_output( 1.0e-6 , 1.0e-6 , stepper_type() );
```
### Table 7. Generation functions make_controlled( abs_error , rel_error , stepper )

<table>
<thead>
<tr>
<th>Stepper</th>
<th>Result of make_controlled</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>runge_kutta_cash_karp54</td>
<td>controlled_runge_kutta&lt; runge_kutta_cash_karp54 , default_error_checker... &gt;</td>
<td>$a_s = 1$, $a_{ddt} = 1$</td>
</tr>
<tr>
<td>runge_kutta_fehlberg78</td>
<td>controlled_runge_kutta&lt; runge_kutta_fehlberg78 , default_error_checker... &gt;</td>
<td>$a_s = 1$, $a_{ddt} = 1$</td>
</tr>
<tr>
<td>runge_kutta_dopri5</td>
<td>controlled_runge_kutta&lt; runge_kutta_dopri5 , default_error_checker... &gt;</td>
<td>$a_s = 1$, $a_{ddt} = 1$</td>
</tr>
<tr>
<td>rosenbrock4</td>
<td>rosenbrock4_controlled&lt; rosenbrock4 &gt;</td>
<td>-</td>
</tr>
</tbody>
</table>

### Table 8. Generation functions make_dense_output( abs_error , rel_error , stepper )

<table>
<thead>
<tr>
<th>Stepper</th>
<th>Result of make_dense_output</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>runge_kutta_dopri5</td>
<td>dense_output_runge_kutta&lt; controlled_runge_kutta&lt; runge_kutta_dopri5 , default_error_checker... &gt; &gt;</td>
<td>$a_s = 1$, $a_{ddt} = 1$</td>
</tr>
<tr>
<td>rosenbrock4</td>
<td>rosenbrock4_dense_output&lt; rosenbrock4_controller&lt; rosenbrock4 &gt; &gt;</td>
<td>-</td>
</tr>
</tbody>
</table>

### Integrate functions

Integrate functions perform the time evolution of a given ODE from some starting time $t_0$ to a given end time $t_1$ and starting at state $x_0$ by subsequent calls of a given stepper's `do_step` function. Additionally, the user can provide an `__observer` to analyze the state during time evolution. There are five different integrate functions which have different strategies on when to call the observer function during integration. All of the integrate functions except `integrate_n_steps` can be called with any stepper following one of the stepper concepts: Stepper, Error Stepper, Controlled Stepper, Dense Output Stepper. Depending on the abilities of the stepper, the integrate functions make use of step-size control or dense output.

### Equidistant observer calls

If observer calls at equidistant time intervals $dt$ are needed, the `integrate_const` or `integrate_n_steps` function should be used. We start with explaining `integrate_const`:

```cpp
integrate_const( stepper , system , x0 , t0 , t1 , dt )
```

```cpp
integrate_const( stepper , system , x0 , t0 , t1 , dt , observer )
```

These integrate the ODE given by `system` with subsequent steps from `stepper`. Integration start at $t_0$ and $x_0$ and ends at some $t' = t_0 + n \ dt$ with $n$ such that $t_f - \ dt < t' \leq t_f$, $x_0$ is changed to the approximative solution $x(t')$ at the end of integration. If provided, the observer is invoked at times $t_0, t_0 + \ dt, t_0 + 2\ dt, \ldots, t'$. `integrate_const` returns the number of steps performed during the integration. Note that if you are using a simple Stepper or Error Stepper and want to make exactly $n$ steps you should prefer the `integrate_n_steps` function below.
• If \texttt{stepper} is a \texttt{Stepper} or \texttt{Error Stepper} then \texttt{dt} is also the step size used for integration and the observer is called just after every step.

• If \texttt{stepper} is a \texttt{Controlled Stepper} then \texttt{dt} is the initial step size. The actual step size will change due to error control during time evolution. However, if an observer is provided the step size will be adjusted such that the algorithm always calculates \(x(t)\) at \(t = t_0 + n \, dt\) and calls the observer at that point. Note that the use of \texttt{Controlled Stepper} is reasonable here only if \texttt{dt} is considerably larger than typical step sizes used by the stepper.

• If \texttt{stepper} is a \texttt{Dense Output Stepper} then \texttt{dt} is the initial step size. The actual step size will be adjusted during integration due to error control. If an observer is provided dense output is used to calculate \(x(t)\) at \(t = t_0 + n \, dt\).

\textbf{Integrate a given number of steps}

This function is very similar to \texttt{integrate\_const} above. The only difference is that it does not take the end time as parameter, but rather the number of steps. The integration is then performed until the time \(t_0+n\times dt\).

\begin{verbatim}
integrate\_n\_steps( stepper , system , x0 , t0 , dt , n )
integrate\_n\_steps( stepper , system , x0 , t0 , dt , n , observer )
\end{verbatim}

Integrates the ODE given by \texttt{system} with subsequent steps from \texttt{stepper} starting at \(x_0\) and \(t_0\). If provided, \texttt{observer} is called after every step and at the beginning with \(t_0\), similar as above. The approximate result for \(x(t)\) at \(t = t_0 + n \, dt\) is stored in \texttt{x0}. This function returns the end time \(t_0 + n\times dt\).

\textbf{Observer calls at each step}

If the observer should be called at each time step then the \texttt{integrate\_adaptive} function should be used. Note that in the case of \texttt{Controlled Stepper} or \texttt{Dense Output Stepper} this leads to non-equidistant observer calls as the step size changes.

\begin{verbatim}
integrate\_adaptive( stepper , system , x0 , t0 , t1 , dt )
integrate\_adaptive( stepper , system , x0 , t0 , t1 , dt , observer )
\end{verbatim}

Integrates the ODE given by \texttt{system} with subsequent steps from \texttt{stepper}. Integration start at \(t_0\) and \(x_0\) and ends at \(t_1\). \texttt{x0} is changed to the approximative solution \(x(t_f)\) at the end of integration. If provided, the \texttt{observer} is called after each step (and before the first step at \(t_0\)). \texttt{integrate\_adaptive} returns the number of steps performed during the integration.

• If \texttt{stepper} is a \texttt{Stepper} or \texttt{Error Stepper} then \texttt{dt} is the step size used for integration and \texttt{integrate\_adaptive} behaves like \texttt{integrate\_const} except that for the last step the step size is reduced to ensure we end exactly at \(t_1\). If provided, the observer is called at each step.

• If \texttt{stepper} is a \texttt{Controlled Stepper} then \texttt{dt} is the initial step size. The actual step size is changed according to error control of the stepper. For the last step, the step size will be reduced to ensure we end exactly at \(t_1\). If provided, the observer is called after each time step (and before the first step at \(t_0\)).

• If \texttt{stepper} is a \texttt{Dense Output Stepper} then \texttt{dt} is the initial step size and \texttt{integrate\_adaptive} behaves just like for \texttt{Controlled Stepper} above. No dense output is used.

\textbf{Observer calls at given time points}

If the observer should be called at some user given time points the \texttt{integrate\_times} function should be used. The times for observer calls are provided as a sequence of time values. The sequence is either defined via two iterators pointing to begin and end of the sequence or in terms of a \texttt{Boost.Range} object.

\begin{verbatim}
integrate\_times( stepper , system , x0 , times\_start , times\_end , dt , observer )
integrate\_times( stepper , system , x0 , time\_range , dt , observer )
\end{verbatim}

Integrates the ODE given by \texttt{system} with subsequent steps from \texttt{stepper}. Integration starts at \(*times\_start\) and ends exactly at \(*times\_end-1\). \texttt{x0} contains the approximate solution at the end point of integration. This function requires an observer which
is invoked at the subsequent times \( \times_{\text{start}}++ \) until \( \times_{\text{start}} == \times_{\text{end}} \). If called with a Boost.Range \( \times_{\text{range}} \) the function behaves the same with \( \times_{\text{start}} = \text{boost::begin}( \times_{\text{range}} ) \) and \( \times_{\text{end}} = \text{boost::end}( \times_{\text{range}} ) \). \( \text{integrate\_times} \) returns the number of steps performed during the integration.

- If stepper is a Stepper or Error Stepper \( \delta_t \) is the step size used for integration. However, whenever a time point from the sequence is approached the step size \( \delta_t \) will be reduced to obtain the state \( x(t) \) exactly at the time point.
- If stepper is a Controlled Stepper then \( \delta_t \) is the initial step size. The actual step size is adjusted during integration according to error control. However, if a time point from the sequence is approached the step size is reduced to obtain the state \( x(t) \) exactly at the time point.
- If stepper is a Dense Output Stepper then \( \delta_t \) is the initial step size. The actual step size is adjusted during integration according to error control. Dense output is used to obtain the states \( x(t) \) at the time points from the sequence.

**Convenience integrate function**

Additionally to the sophisticated integrate function above odeint also provides a simple integrate routine which uses a dense output stepper based on runge_kutta_dopri5 with standard error bounds \( 10^{-6} \) for the steps.

\[
\text{integrate( system , x0 , t0 , t1 , dt )}
\]
\[
\text{integrate( system , x0 , t0 , t1 , dt , observer )}
\]

This function behaves exactly like integrate_adaptive above but no stepper has to be provided. It also returns the number of steps performed during the integration.

**State types, algebras and operations**

In odeint the stepper algorithms are implemented independently of the underlying fundamental mathematical operations. This is realized by giving the user full control over the state type and the mathematical operations for this state type. Technically, this is done by introducing three concepts: StateType, Algebra, Operations. Most of the steppers in odeint expect three class types fulfilling these concepts as template parameters. Note that these concepts are not fully independent of each other but rather a valid combination must be provided in order to make the steppers work. In the following we will give some examples on reasonable state_type-algebra-operations combinations. For the most common state types, like vector<double> or array<double,N> the default values range_algebra and default_operations are perfectly fine and odeint can be used as is without worrying about algebra/operations at all.

**Important**

state_type, algebra and operations are not independent, a valid combination must be provided to make odeint work properly

Moreover, as odeint handles the memory required for intermediate temporary objects itself, it also needs knowledge about how to create state_type objects and maybe how to allocate memory (resizing). All in all, the following things have to be taken care of when odeint is used with non-standard state types:

- construction/destruction
- resizing (if possible/required)
- algebraic operations

Again, odeint already provides basic interfaces for most of the usual state types. So if you use a std::vector, or a boost::array as state type no additional work is required, they just work out of the box.
**Construction/Resizing**

We distinguish between two basic state types: fixed sized and dynamically sized. For fixed size state types the default constructor `state_type()` already allocates the required memory, prominent example is `boost::array<T,N>`. Dynamically sized types have to be resized to make sure enough memory is allocated, the standard constructor does not take care of the resizing. Examples for this are the STL containers like `vector<double>`.

The most easy way of getting your own state type to work with odeint is to use a fixed size state, base calculations on the `range_algebra` and provide the following functionality:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct State</td>
<td>State x()</td>
<td>void</td>
<td>Creates an instance of State and allocates memory.</td>
</tr>
<tr>
<td>Begin of the sequence</td>
<td>boost::begin(x)</td>
<td>Iterator</td>
<td>Returns an iterator pointing to the begin of the sequence</td>
</tr>
<tr>
<td>End of the sequence</td>
<td>boost::end(x)</td>
<td>Iterator</td>
<td>Returns an iterator pointing to the end of the sequence</td>
</tr>
</tbody>
</table>

**Warning**

If your state type does not allocate memory by default construction, you **must define it as resizeable** and provide resize functionality (see below). Otherwise segmentation faults will occur.

So fixed sized arrays supported by Boost.Range immediately work with odeint. For dynamically sized arrays one has to additionally supply the resize functionality. First, the state has to be tagged as resizeable by specializing the struct `is_resizeable` which consists of one typedef and one bool value:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resizability</td>
<td><code>is_resizeable&lt;State&gt;::type</code></td>
<td><code>boost::true_type</code> or <code>boost::false_type</code></td>
<td>Determines resizeability of the state type, returns <code>boost::true_type</code> if the state is resizeable.</td>
</tr>
<tr>
<td>Resizability</td>
<td><code>is_resizeable&lt;State&gt;::value</code></td>
<td><code>bool</code></td>
<td>Same as above, but with bool value.</td>
</tr>
</tbody>
</table>

Defining `type` to be `true_type` and `value` as `true` tells odeint that your state is resizeable. By default, odeint now expects the support of `boost::size(x)` and `x.resize( boost::size(y) )` member function for resizing:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get size</td>
<td><code>boost::size( x )</code></td>
<td><code>size_type</code></td>
<td>Returns the current size of x.</td>
</tr>
<tr>
<td>Resize</td>
<td><code>x.resize( boost::size( y ) )</code></td>
<td><code>void</code></td>
<td>Resizes x to have the same size as y.</td>
</tr>
</tbody>
</table>

**Using the container interface**

As a first example we take the most simple case and implement our own vector `my_vector` which will provide a container interface. This makes Boost.Range working out-of-box. We add a little functionality to our vector which makes it allocate some default capacity by construction. This is helpful when using resizing as then a resize can be assured to not require a new allocation.
template< int MAX_N >
class my_vector
{
    typedef std::vector< double > vector;

public:
    typedef vector::iterator iterator;
    typedef vector::const_iterator const_iterator;

public:
    my_vector( const size_t N ) :
    m_v( N )
    {
        m_v.reserve( MAX_N );
    }

    my_vector() :
    m_v()
    {
        m_v.reserve( MAX_N );
    }

    // ... [ implement container interface ]

The only thing that has to be done other than defining is thus declaring my_vector as resizeable:

// define my_vector as resizeable
namespace boost { namespace numeric { namespace odeint {
    template<size_t N>
    struct is_resizeable< my_vector<N> >
    {
        typedef boost::true_type type;
        static const bool value = type::value;
    };

}} }

If we wouldn't specialize the is_resizeable template, the code would still compile but odeint would not adjust the size of temporary internal instances of my_vector and hence try to fill zero-sized vectors resulting in segmentation faults! The full example can be found in my_vector.cpp

std::list

If your state type does work with Boost.Range, but handles resizing differently you are required to specialize two implementations used by odeint to check a state's size and to resize:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check size</td>
<td>same_size_impl&lt;State,State&gt;::same_size(x, y)</td>
<td>bool</td>
<td>Returns true if the size of x equals the size of y.</td>
</tr>
<tr>
<td>Resize</td>
<td>resize_impl&lt;State,State&gt;::resize(x, y)</td>
<td>void</td>
<td>Resizes x to have the same size as y.</td>
</tr>
</tbody>
</table>
As an example we will use a std::list as state type in odeint. Because std::list is not supported by boost::size we have to replace the same_size and resize implementation to get list to work with odeint. The following code shows the required template specializations:

```cpp
typedef std::list< double > state_type;

namespace boost { namespace numeric { namespace odeint {

  template< >
  struct is_resizeable< state_type >
  { // declare resizeability
    typedef boost::true_type type;
    const static bool value = type::value;
  };

  template< >
  struct same_size_impl< state_type , state_type >
  { // define how to check size
    static bool same_size( const state_type &v1 ,
                          const state_type &v2 )
    {
      return v1.size() == v2.size();
    }
  };

  template< >
  struct resize_impl< state_type , state_type >
  { // define how to resize
    static void resize( state_type &v1 ,
                        const state_type &v2 )
    {
      v1.resize( v2.size() );
    }
  };

}}}
```

With these definitions odeint knows how to resize std::lists and so they can be used as state types. A complete example can be found in list_lattice.cpp.

### Algebras and Operations

To provide maximum flexibility odeint is implemented in a highly modularized way. This means it is possible to change the underlying mathematical operations without touching the integration algorithms. The fundamental mathematical operations are those of a vector space, that is addition of state_types and multiplication of state_types with a scalar (time_type). In odeint this is realized in two concepts: Algebra and Operations. The standard way how this works is by the range algebra which provides functions that apply a specific operation to each of the individual elements of a container based on the Boost.Range library. If your state type is not supported by Boost.Range there are several possibilities to tell odeint how to do algebraic operations:

- Implement boost::begin and boost::end for your state type so it works with Boost.Range.
- Implement vector-vector addition operator + and scalar-vector multiplication operator * and use the non-standard vector_space_algebra.
- Implement your own algebra that implements the required functions.

### GSL Vector

In the following example we will try to use the gsl_vector type from GSL (GNU Scientific Library) as state type in odeint. We will realize this by implementing a wrapper around the gsl_vector that takes care of construction/destruction. Also, Boost.Range is extended such that it works with gsl_vectors as well which required also the implementation of a new gsl_iterator.
The GSL is a C library, so \texttt{gsl\_vector} has neither constructor, nor destructor or any \texttt{begin} or \texttt{end} function, no iterators at all. So to make it work with odeint plenty of things have to be implemented. Note that all of the work shown here is already included in odeint, so using \texttt{gsl\_vectors} in odeint doesn’t require any further adjustments. We present it here just as an educational example.

We start with defining appropriate constructors and destructors. This is done by specializing the \texttt{state\_wrapper} for \texttt{gsl\_vector}. State wrappers are used by the steppers internally to create and manage temporary instances of state types:

```cpp
template<>
struct state_wrapper< gsl_vector* >
{
    typedef double value_type;
    typedef gsl_vector* state_type;
    typedef state_wrapper< gsl_vector* > state_wrapper_type;

    state_type m_v;

    state_wrapper( )
    {
        m_v = gsl_vector_alloc( 1 );
    }

    state_wrapper( const state_wrapper_type &x )
    {
        resize( m_v , x.m_v );
        gsl_vector_memcpy( m_v , x.m_v );
    }

    ~state_wrapper()
    {
        gsl_vector_free( m_v );
    }
};
```

This \texttt{state\_wrapper} specialization tells odeint how \texttt{gsl\_vectors} are created, copied and destroyed. Next we need resizing, this is required because \texttt{gsl\_vectors} are dynamically sized objects:
template<>
struct is_resizeable< gsl_vector* >
{
    typedef boost::true_type type;
    const static bool value = type::value;
};

template <>
struct same_size_impl< gsl_vector* , gsl_vector* >
{
    static bool same_size( const gsl_vector* x , const gsl_vector* y )
    {
        return x->size == y->size;
    }
};

template <>
struct resize_impl< gsl_vector* , gsl_vector* >
{
    static void resize( gsl_vector* x , const gsl_vector* y )
    {
        gsl_vector_free( x );
        x = gsl_vector_alloc( y->size );
    }
};

Up to now, we defined creation/destruction and resizing, but gsl_vectors also don't support iterators, so we first implement a gsl iterator:

/*
 * defines an iterator for gsl_vector
 */
class gsl_vector_iterator
    : public boost::iterator_facade< gsl_vector_iterator , double ,
                                   boost::random_access_traversal_tag >
{
    public :

        gsl_vector_iterator( void ): m_p(0) , m_stride( 0 ) { }
    explicit gsl_vector_iterator( gsl_vector *p ) : m_p( p->data ) , m_stride( p->stride ) { }
    friend gsl_vector_iterator end_iterator( gsl_vector *);

    private :

        friend class boost::iterator_core_access;
        friend class const_gsl_vector_iterator;

        void increment( void ) { m_p += m_stride; }
    void decrement( void ) { m_p -= m_stride; }
    void advance( ptrdiff_t n ) { m_p += n*m_stride; }
    bool equal( const gsl_vector_iterator &other ) const { return this->m_p == other.m_p; }
    bool equal( const const_gsl_vector_iterator &other ) const;
    double& dereference( void ) const { return *(m_p); }

    double *m_p;
    size_t m_stride;
};

A similar class exists for the const version of the iterator. Then we have a function returning the end iterator (similarly for const again):
gsl_vector_iterator end_iterator( gsl_vector *x )
{
  gsl_vector_iterator iter( x );
  iter.m_p += iter.m_stride * x->size;
  return iter;
}

Finally, the bindings for Boost.Range are added:

```cpp
// template<>
inline gsl_vector_iterator range_begin( gsl_vector *x )
{
  return gsl_vector_iterator( x );
}

// template<>
inline gsl_vector_iterator range_end( gsl_vector *x )
{
  return end_iterator( x );
}
```

Again with similar definitions for the `const` versions. This eventually makes odeint work with gsl vectors as state types. The full code for these bindings is found in `gsl_wrapper.hpp`. It might look rather complicated but keep in mind that gsl is a pre-compiled C library.

**Vector Space Algebra**

As seen above, the standard way of performing algebraic operations on container-like state types in odeint is to iterate through the elements of the container and perform the operations element-wise on the underlying value type. This is realized by means of the `range_algebra` that uses Boost.Range for obtaining iterators of the state types. However, there are other ways to implement the algebraic operations on containers, one of which is defining the addition/multiplication operators for the containers directly and then using the `vector_space_algebra`. If you use this algebra, the following operators have to be defined for the `state_type`:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>x + y</td>
<td>state_type</td>
<td>Calculates the vector sum 'x+y'.</td>
</tr>
<tr>
<td>Assign addition</td>
<td>x += y</td>
<td>state_type</td>
<td>Performs x+y in place.</td>
</tr>
<tr>
<td>Scalar multiplication</td>
<td>a * x</td>
<td>state_type</td>
<td>Performs multiplication of vector x with scalar a.</td>
</tr>
<tr>
<td>Assign scalar multiplication</td>
<td>x *= a</td>
<td>state_type</td>
<td>Performs in-place multiplication of vector x with scalar a.</td>
</tr>
</tbody>
</table>

Defining these operators makes your state type work with any basic Runge-Kutta stepper. However, if you want to use step-size control, some more functionality is required. Specifically, operations like \( \max_i \left( \frac{|err_i|}{\alpha \cdot |s_i|} \right) \) have to be performed. \( err \) and \( s \) are state_types, \( \alpha \) is a scalar. As you can see, we need element wise absolute value and division as well as a reduce operation to get the maximum value. So for controlled steppers the following things have to be implemented:
### Boost.Ublas

As an example for the employment of the `vector_space_algebra` we will adopt `ublas::vector` from `Boost.uBLAS` to work as a state type in `odeint`. This is particularly easy because `ublas::vector` supports vector-vector addition and scalar-vector multiplication described above as well as `boost::size`. It also has a resize member function so all that has to be done in this case is to declare resizability:

```cpp
typedef boost::numeric::ublas::vector< double > state_type;

namespace boost { namespace numeric { namespace odeint {
    template<>
    struct is_resizeable< state_type >
    {
        typedef boost::true_type type;
        const static bool value = type::value;
    };
}}}
```

Now `ublas::vector` can be used as state type for simple Runge-Kutta steppers in `odeint` by specifying the `vector_space_algebra` as algebra in the template parameter list of the stepper. The following code shows the corresponding definitions:

```cpp
int main()
{
    state_type x(3);
    x[0] = 10.0; x[1] = 5.0 ; x[2] = 0.0;
    typedef runge_kutta4< state_type , double , state_type , double , vector_space_algebra > step_per;
    integrate_const( stepper() , lorenz , x ,
        0.0 , 10.0 , 0.1 );
}
```

Note again, that we haven't supported the requirements for controlled steppers, but only for simple Runge-Kutta methods. You can find the full example in `lorenz_ublas.cpp`.

---

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Division</td>
<td>x / y</td>
<td><code>state_type</code></td>
<td>Calculates the element-wise division 'x/y'</td>
</tr>
<tr>
<td>Absolute value</td>
<td><code>abs(x)</code></td>
<td><code>state_type</code></td>
<td>Element wise absolute value</td>
</tr>
<tr>
<td>Reduce</td>
<td><code>vector_space_reduce_impl&lt; state_type &gt;::reduce(state , operation , init)</code></td>
<td><code>value_type</code></td>
<td>Performs the operation for subsequently each element of state and returns the aggregate value. E.g.</td>
</tr>
</tbody>
</table>

| init = operator( init , state[0] ); |
| init = operator( init , state[1] ) |
| ... |

---

**Boost.Numeric.Odeint**

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Point type

Here we show how to implement the required operators on a state type. As example we define a new class `point3D` representing a three-dimensional vector with components x,y,z and define addition and scalar multiplication operators for it. We use Boost.Operators to reduce the amount of code to be written. The class for the point type looks as follows:

```cpp
class point3D {
    boost::additive1< point3D ,
    boost::additive2< point3D , double ,
    boost::multiplicative2< point3D , double > > >

public:

double x , y , z;

point3D()
    : x( 0.0 ) , y( 0.0 ) , z( 0.0 )
{ }

point3D( const double val )
    : x( val ) , y( val ) , z( val )
{ }

point3D( const double _x , const double _y , const double _z )
    : x( _x ) , y( _y ) , z( _z )
{ }

point3D& operator+=( const point3D &p )
{
    x += p.x; y += p.y; z += p.z;
    return *this;
}

point3D& operator*=( const double a )
{
    x *= a; y *= a; z *= a;
    return *this;
}
}
```

By deriving from Boost.Operators classes we don't have to define outer class operators like `operator+( point3D , point3D )` because that is taken care of by the operators library. Note that for simple Runge-Kutta schemes (like `runge_kutta4`) only the `+` and `*` operators are required. If, however, a controlled stepper is used one also needs to specify the division operator `/` because calculation of the error term involves an element wise division of the state types. Additionally, controlled steppers require an `abs` function calculating the element-wise absolute value for the state type:

```cpp
// only required for steppers with error control
point3D operator/( const point3D &p1 , const point3D &p2 )
{
    return point3D( p1.x/p2.x , p1.y/p2.y , p1.z/p1.z );
}

point3D abs( const point3D &p )
{
    return point3D( std::abs(p.x) , std::abs(p.y) , std::abs(p.z) );
}
```

Finally, we have to add a specialization for `reduce` implementing a reduction over the state type:
namespace boost { namespace numeric { namespace odeint {

// specialization of vector_space_reduce, only required for steppers with error control

template<>
struct vector_space_reduce< point3D >
{
    template< class Value , class Op >
    Value operator() ( const point3D &p , Op op , Value init )
    {
        init = op( init , p.x );
        //std::cout << init << " ";
        init = op( init , p.y );
        //std::cout << init << " ";
        init = op( init , p.z );
        //std::cout << init << std::endl;
        return init;
    }
};
}
}

Again, note that the two last steps were only required if you want to use controlled steppers. For simple steppers definition of the simple += and *= operators are sufficient. Having defined such a point type, we can easily perform the integration on a Lorenz system by using the vector_space_algebra again:

const double sigma = 10.0;
const double R = 28.0;
const double b = 8.0 / 3.0;

void lorenz( const point3D &x , point3D &dxdt , const double t )
{
    dxdt.x = sigma * ( x.y - x.x );
    dxdt.y = R * x.x - x.y - x.x * x.z;
    dxdt.z = -b * x.z + x.x * x.y;
}

using namespace boost::numeric::odeint;

int main()
{
    point3D x( 10.0 , 5.0 , 5.0 );
    // point type defines it's own operators -> use vector_space_algebra !
    typedef runge_kutta_dopri5< point3D , double , point3D ,
    double , vector_space_algebra > stepper;
    int steps = integrate_adaptive( make_controlled<stepper>( 1E-10 , 1E-10 ) , lorenz , x ,
    0.0 , 10.0 , 0.1 );
    std::cout << x << std::endl;
    std::cout << "steps: " << steps << std::endl;
}

The whole example can be found in lorenz_point.cpp

gsl_vector, gsl_matrix, ublas::matrix, blitz::matrix, thrust

Adapt your own operations

to be continued

• thrust

• gsl_complex
Using boost::ref

In odeint all system functions and observers are passed by value. For example, if you call a do_step method of a particular stepper or the integration functions, your system and your stepper will be passed by value:

```cpp
rk4.do_step( sys , x , t , dt ); // pass sys by value
```

This behavior is suitable for most systems, especially if your system does not contain any data or only a few parameters. However, in some cases you might contain some large amount of data with you system function and passing them by value is not desired since the data would be copied.

In such cases you can easily use boost::ref (and its relative boost::cref) which passes its argument by reference (or constant reference). odeint will unpack the arguments and no copying at all of your system object will take place:

```cpp
rk4.do_step( boost::ref( sys ) , x , t , dt ); // pass sys as references
```

The same mechanism can be used for the observers in the integrate functions.

Tip

If you are using C++11 you can also use std::ref and std::cref

Using boost::range

Most steppers in odeint also accept the state give as a range. A range is sequence of values modeled by a range concept. See Boost.Range for an overview over existing concepts and examples of ranges. This means that the state_type of the stepper need not necessarily be used to call the do_step method.

One use-case for Boost.Range in odeint has been shown in Chaotic System where the state consists of two parts: one for the original system and one for the perturbations. The ranges are used to initialize (solve) only the system part where the perturbation part is not touched, that is a range consisting only of the system part is used. After that the complete state including the perturbations is solved.

Another use case is a system consisting of coupled units where you want to initialize each unit separately with the ODE of the uncoupled unit. An example is a chain of coupled van-der-Pol-oscillators which are initialized uniformly from the uncoupled van-der-Pol-oscillator. Then you can use Boost.Range to solve only one individual oscillator in the chain.

In short, you can Boost.Range to use one state within two system functions which expect states with different sizes.

An example was given in the Chaotic System tutorial. Using Boost.Range usually means that your system function needs to adapt to the iterators of Boost.Range. That is, your function is called with a range and you need to get the iterators from that range. This can easily be done. You have to implement your system as a class or a struct and you have to templatize the operator(). Then you can use the range_iterator-meta function and boost::begin and boost::end to obtain the iterators of your range:
class sys
{
    template< class State , class Deriv >
    void operator()( const State &x_ , Deriv &dxdt_ , double t ) const
    {
        typename boost::range_iterator< const State >::type x = boost::begin( x_ );
        typename boost::range_iterator< Deriv >::type dxdt = boost::begin( dxdt_ );

        // fill dxdt
    }
};

If your range is a random access-range you can also apply the bracket operator to the iterator to access the elements in the range:

class sys
{
    template< class State , class Deriv >
    void operator()( const State &x_ , Deriv &dxdt_ , double t ) const
    {
        typename boost::range_iterator< const State >::type x = boost::begin( x_ );
        typename boost::range_iterator< Deriv >::type dxdt = boost::begin( dxdt_ );

        dxdt[0] = f1( x[0] , x[1] );
        dxdt[1] = f2( x[0] , x[1] );
    }
};

The following two tables show which steppers and which algebras are compatible with Boost.Range.
### Table 9. Steppers supporting Boost.Range

<table>
<thead>
<tr>
<th>Stepper</th>
</tr>
</thead>
<tbody>
<tr>
<td>adams_bashforth_moulton</td>
</tr>
<tr>
<td>bulirsch_stoer_dense_out</td>
</tr>
<tr>
<td>bulirsch_stoer</td>
</tr>
<tr>
<td>controlled_runge_kutta</td>
</tr>
<tr>
<td>dense_output_runge_kutta</td>
</tr>
<tr>
<td>euler</td>
</tr>
<tr>
<td>explicit_error_generic_rk</td>
</tr>
<tr>
<td>explicit_generic_rk</td>
</tr>
<tr>
<td>rosenbrock4_controller</td>
</tr>
<tr>
<td>rosenbrock4_dense_output</td>
</tr>
<tr>
<td>rosenbrock4</td>
</tr>
<tr>
<td>runge_kutta4_classic</td>
</tr>
<tr>
<td>runge_kutta4</td>
</tr>
<tr>
<td>runge_kutta_cash_karp54_classic</td>
</tr>
<tr>
<td>runge_kutta_cash_karp54</td>
</tr>
<tr>
<td>runge_kutta_dopri5</td>
</tr>
<tr>
<td>runge_kutta_fehlberg78</td>
</tr>
<tr>
<td>symplectic_euler</td>
</tr>
<tr>
<td>symplectic_rkn_sb3a_mclachlan</td>
</tr>
</tbody>
</table>

### Table 10. Algebras supporting Boost.Range

<table>
<thead>
<tr>
<th>algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>range_algebra</td>
</tr>
<tr>
<td>thrust_algebra</td>
</tr>
</tbody>
</table>

### Binding member functions

Binding member functions to a function objects suitable for odeint system function is not easy, at least in C++03. The usual way of using __boost_bind does not work because of the forwarding problem. odeint provides two do_step method which only differ in the const specifiers of the arguments and __boost_bind binders only provide the specializations up to two argument which is not enough for odeint.
But one can easily implement the according binders themself:

```cpp
template< class Obj , class Mem >
class ode_wrapper
{
    Obj m_obj;
    Mem m_mem;

gnpublic:

    ode_wrapper( Obj obj , Mem mem ) : m_obj( obj ) , m_mem( mem ) { }

    template< class State , class Deriv , class Time >
    void operator()( const State &x , Deriv &dxdt , Time t )
    {
        (m_obj.*m_mem)( x , dxdt , t );
    }

};

template< class Obj , class Mem >
ode_wrapper< Obj , Mem > make_ode_wrapper( Obj obj , Mem mem )
{
    return ode_wrapper< Obj , Mem >( obj , mem );
}

One can use this binder as follows

```cpp
struct lorenz
{
    void ode( const state_type &x , state_type &dxdt , double t ) const
    {
        dxdt[0] = 10.0 * ( x[1] - x[0] );
        dxdt[1] = 28.0 * x[0] - x[1] - x[0] * x[2];
        dxdt[2] = -8.0 / 3.0 * x[2] + x[0] * x[1];
    }
};

int main( int argc , char *argv[] )
{
    using namespace boost::numeric::odeint;
    state_type x = {{ 10.0 , 10.0 , 10.0 }};
    integrate_const( runge_kutta4< state_type >() , make_ode_wrapper( lorenz() , &lorenz::ode ) ,
        x , 0.0 , 10.0 , 0.01 );
    return 0;
}

Binding member functions in C++11

In C++11 one can use std::bind and one does not need to implement the bind themself:

```cpp
namespace pl = std::placeholders;

state_type x = {{ 10.0 , 10.0 , 10.0 }};
integrate_const( runge_kutta4< state_type >() ,
    std::bind( &lorenz::ode , lorenz() , pl::_1 , pl::_2 , pl::_3 ) ,
    x , 0.0 , 10.0 , 0.01 );

```
Concepts

System

Description

The System concept models the algorithmic implementation of the rhs. of the ODE \( x' = f(x,t) \). The only requirement for this concept is that it should be callable with a specific parameter syntax (see below). A System is typically implemented as a function or a functor. Systems fulfilling this concept are required by all Runge-Kutta steppers as well as the Bulirsch-Stoer steppers. However, symplectic and implicit steppers work with other system concepts, see Symplectic System and Implicit System.

Notation

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>A type representing the state ( x ) of the ODE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deriv</td>
<td>A type representing the derivative ( x' ) of the ODE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time</td>
<td>A type representing the time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sys</td>
<td>An object of type System</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>Object of type State</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dxdt</td>
<td>Object of type Deriv</td>
<td></td>
<td></td>
</tr>
<tr>
<td>t</td>
<td>Object of type Time</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Valid expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sys (x, dxdt, t)</td>
<td>void</td>
<td>Calculates ( f(x,t) ), the result is stored into dxdt</td>
</tr>
</tbody>
</table>

Symplectic System

Description

This concept describes how to define a symplectic system written with generalized coordinate \( q \) and generalized momentum \( p \):

\[
q'(t) = f(p) \\
p'(t) = g(q)
\]

Such a situation is typically found for Hamiltonian systems with a separable Hamiltonian:

\[
H(p,q) = H_{\text{kin}}(p) + V(q)
\]

which gives the equations of motion:

\[
q'(t) = dH_{\text{kin}} / dp = f(p) \\
p'(t) = dV / dq = g(q)
\]
The algorithmic implementation of this situation is described by a pair of callable objects for \( f \) and \( g \) with a specific parameter signature. Such a system should be implemented as a std::pair of functions or a functors. Symplectic systems are used in symplectic steppers like `symplectic_rkn_sb3a_mclachlan`.

### Notation

<table>
<thead>
<tr>
<th>System</th>
<th>A type that is a model of SymplecticSystem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coor</td>
<td>The type of the coordinate ( q )</td>
</tr>
<tr>
<td>Momentum</td>
<td>The type of the momentum ( p )</td>
</tr>
<tr>
<td>CoorDeriv</td>
<td>The type of the derivative of coordinate ( q' )</td>
</tr>
<tr>
<td>MomentumDeriv</td>
<td>The type of the derivative of momentum ( p' )</td>
</tr>
<tr>
<td>sys</td>
<td>An object of the type System</td>
</tr>
<tr>
<td>q</td>
<td>Object of type Coor</td>
</tr>
<tr>
<td>p</td>
<td>Object of type Momentum</td>
</tr>
<tr>
<td>dqdt</td>
<td>Object of type CoorDeriv</td>
</tr>
<tr>
<td>dpdt</td>
<td>Object of type MomentumDeriv</td>
</tr>
</tbody>
</table>

### Valid expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check for pair</td>
<td>boost::is_pair&lt; System, true_ctype&gt;</td>
<td>boost::true_ctype</td>
<td>Check if System is a pair</td>
</tr>
<tr>
<td>Calculate ( dq/dt = f(p) )</td>
<td>sys.first( p, dqdt )</td>
<td>void</td>
<td>Calculates ( f(p) ), the result is stored into ( dqdt )</td>
</tr>
<tr>
<td>Calculate ( dp/dt = g(q) )</td>
<td>sys.second( q, dpdt )</td>
<td>void</td>
<td>Calculates ( g(q) ), the result is stored into ( dpdt )</td>
</tr>
</tbody>
</table>

### Simple Symplectic System

### Description

In most Hamiltonian systems the kinetic term is a quadratic term in the momentum \( H_{\text{kin}} = p^2 / 2m \) and in many cases it is possible to rescale coordinates and set \( m = 1 \) which leads to a trivial equation of motion:

\[
q'(t) = f(p) = p.
\]

while for \( p' \) we still have the general form

\[
p'(t) = g(q)
\]

As this case is very frequent we introduced a concept where only the nontrivial equation for \( p' \) has to be provided to the symplectic stepper. We call this concept `SimpleSymplecticSystem`.

### Notation

<table>
<thead>
<tr>
<th>System</th>
<th>A type that is a model of SimpleSymplecticSystem</th>
</tr>
</thead>
</table>
Coor The type of the coordinate q

MomentumDeriv The type of the derivative of momentum p'

sys An object that models System

q Object of type Coor

dpdt Object of type MomentumDeriv

Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Check for pair</td>
<td>boost::is_pair&lt; System &gt;::type</td>
<td>boost::mpl::false_</td>
<td>Check if System is a pair, should be evaluated to false in this case.</td>
</tr>
<tr>
<td>Calculate $dp/dt = g(q)$</td>
<td>sys( q , dpdt )</td>
<td>void</td>
<td>Calculates $g(q)$, the result is stored into dpdt</td>
</tr>
</tbody>
</table>

Implicit System

Description

This concept describes how to define a ODE that can be solved by an implicit routine. Implicit routines need not only the function $f(x,t)$ but also the Jacobian $df/dx = A(x,t)$. $A$ is a matrix and implicit routines need to solve the linear problem $Ax = b$. In odeint this is implemented with use of Boost.uBLAS, therefore, the state_type implicit routines is ublas::vector and the matrix is defined as ublas::matrix.

Notation

System A type that is a model of Implicit System

Time A type representing the time of the ODE

sys An object of type System

x Object of type ublas::vector

dxdt Object of type ublas::vector

jacobi Object of type ublas::matrix

t Object of type Time

Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate $dx/dt := f(x,t)$</td>
<td>sys.first( x , dxdt , t )</td>
<td>void</td>
<td>Calculates $f(x,t)$, the result is stored into dxdt</td>
</tr>
<tr>
<td>Calculate $A := df/dx (x,t)$</td>
<td>sys.second( x , jacobi , t )</td>
<td>void</td>
<td>Calculates the Jacobian of $f$ at $x,t$, the result is stored into jacobi</td>
</tr>
</tbody>
</table>
Stepper

This concepts specifies the interface a simple stepper has to fulfill to be used within the integrate functions.

**Description**

The basic stepper concept. A basic stepper following this Stepper concept is able to perform a single step of the solution $x(t)$ of an ODE to obtain $x(t+dt)$ using a given step size $dt$. Basic steppers can be Runge-Kutta steppers, symplectic steppers as well as implicit steppers. Depending on the actual stepper, the ODE is defined as System, Symplectic System, Simple Symplectic System or Implicit System. Note that all error steppers are also basic steppers.

**Refinement of**

- DefaultConstructable
- CopyConstructable

**Associated types**

- **state_type**
  
  Stepper::state_type

  The type characterizing the state of the ODE, hence $x$.

- **deriv_type**
  
  Stepper::deriv_type

  The type characterizing the derivative of the ODE, hence $d x/dt$.

- **time_type**
  
  Stepper::time_type

  The type characterizing the dependent variable of the ODE, hence the time $t$.

- **value_type**
  
  Stepper::value_type

  The numerical data type which is used within the stepper, something like `float`, `double`, `complex<double>`.

- **order_type**
  
  Stepper::order_type

  The type characterizing the order of the ODE, typically `unsigned short`.

- **stepper_category**
  
  Stepper::stepper_category

  A tag type characterizing the category of the stepper. This type must be convertible to `stepper_tag`.

**Notation**

- Stepper A type that is a model of Stepper
- State A type representing the state $x$ of the ODE
A type representing the time \( t \) of the ODE

An object of type \texttt{Stepper}

Object of type \texttt{State}

Objects of type \texttt{Time}

An object defining the ODE. Depending on the Stepper this might be a model of \texttt{System}, \texttt{Symplectic System}, \texttt{Simple Symplectic System} or \texttt{Implicit System}

### Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get the order</td>
<td>\texttt{stepper.order()}</td>
<td>\texttt{order_type}</td>
<td>Returns the order of the stepper.</td>
</tr>
<tr>
<td>Do step</td>
<td>\texttt{stepper.do_step( sys, x, t, dt )}</td>
<td>\texttt{void}</td>
<td>Performs one step of step size ( dt ). The newly obtained state is written in place in ( x ).</td>
</tr>
</tbody>
</table>

### Models

- \texttt{runge_kutta4}
- \texttt{euler}
- \texttt{runge_kutta_cash_karp54}
- \texttt{runge_kutta_dopri5}
- \texttt{runge_kutta_fehlberg78}
- \texttt{modified_midpoint}
- \texttt{rosenbrock4}

### Error Stepper

This concept specifies the interface an error stepper has to fulfill to be used within a \texttt{ControlledErrorStepper}. An error stepper must always fulfill the stepper concept. This can trivially implemented by

```cpp
template< class System >
error_stepper::do_step( System sys , state_type &x , time_type t , time_type dt )
{
    state_type xerr;
    // allocate xerr
    do_step( sys , x , t , dt , xerr );
}
```

### Description

An error stepper following this Error Stepper concept is capable of doing one step of the solution \( x(t) \) of an ODE with step-size \( dt \) to obtain \( x(t+dt) \) and also computing an error estimate \( x_{err} \) of the result. Error Steppers can be Runge-Kutta steppers, symplectic steppers as well as implicit steppers. Based on the stepper type, the ODE is defined as \texttt{System}, \texttt{Symplectic System}, \texttt{Simple Symplectic System} or \texttt{Implicit System}. 
Refinement of

- DefaultConstructable
- CopyConstructable
- Stepper

Associated types

- **state_type**
  ```cpp
  Stepper::state_type
  ```
  The type characterizing the state of the ODE, hence $x$.

- **deriv_type**
  ```cpp
  Stepper::deriv_type
  ```
  The type characterizing the derivative of the ODE, hence $\frac{dx}{dt}$.

- **time_type**
  ```cpp
  Stepper::time_type
  ```
  The type characterizing the dependent variable of the ODE, hence the time $t$.

- **value_type**
  ```cpp
  Stepper::value_type
  ```
  The numerical data type which is used within the stepper, something like `float`, `double`, `complex<double>`.

- **order_type**
  ```cpp
  Stepper::order_type
  ```
  The type characterizing the order of the ODE, typically unsigned short.

- **stepper_category**
  ```cpp
  Stepper::stepper_category
  ```
  A tag type characterizing the category of the stepper. This type must be convertible to `error_stepper_tag`.

Notation

| **ErrorStepper** | A type that is a model of Error Stepper |
| **State**        | A type representing the state $x$ of the ODE |
| **Error**        | A type representing the error calculated by the stepper, usually same as State |
| **Time**         | A type representing the time $t$ of the ODE |
| **stepper**      | An object of type ErrorStepper |
| **x**            | Object of type State |
| **xerr**         | Object of type Error |
Objects of type `Time`

An object defining the ODE, should be a model of either `System`, `Symplectic System`, `Simple Symplectic System` or `Implicit System`.

### Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get the stepper order</td>
<td><code>stepper.order()</code></td>
<td><code>order_type</code></td>
<td>Returns the order of the stepper for one step without error estimation.</td>
</tr>
<tr>
<td>Get the stepper order</td>
<td><code>stepper.stepper_order()</code></td>
<td><code>order_type</code></td>
<td>Returns the order of the stepper for one error estimation step which is used for error calculation.</td>
</tr>
<tr>
<td>Get the error order</td>
<td><code>stepper.errorr_order()</code></td>
<td><code>order_type</code></td>
<td>Returns the order of the error step which is used for error calculation.</td>
</tr>
<tr>
<td>Do step</td>
<td><code>stepper.do_step( sys, x, t, dt )</code></td>
<td><code>void</code></td>
<td>Performs one step of step size <code>dt</code>. The newly obtained state is written in-place to <code>x</code>.</td>
</tr>
<tr>
<td>Do step with error estimation</td>
<td><code>stepper.do_step( sys, x, t, dt, xerr )</code></td>
<td><code>void</code></td>
<td>Performs one step of step size <code>dt</code> with error estimation. The newly obtained state is written in-place to <code>x</code> and the estimated error to <code>xerr</code>.</td>
</tr>
</tbody>
</table>

### Models

- `runge_kutta_cash_karp54`
- `runge_kutta_dopri5`
- `runge_kutta_fehlberg78`
- `rosenbrock4`

### Controlled Stepper

This concept specifies the interface a controlled stepper has to fulfill to be used within `integrate` functions.

### Description

A controlled stepper following this Controlled Stepper concept provides the possibility to perform one step of the solution $x(t)$ of an ODE with step-size `dt` to obtain $x(t+dt)$ with a given step-size `dt`. Depending on an error estimate of the solution the step might be rejected and a smaller step-size is suggested.

### Associated types

- `state_type`

  ```cpp
  Stepper::state_type
  ```
The type characterizing the state of the ODE, hence \( x \).

- **deriv_type**
  
  `Stepper::deriv_type`

  The type characterizing the derivative of the ODE, hence \( d\,x/dt \).

- **time_type**
  
  `Stepper::time_type`

  The type characterizing the dependent variable of the ODE, hence the time \( t \).

- **value_type**
  
  `Stepper::value_type`

  The numerical data type which is used within the stepper, something like `float`, `double`, `complex<double>`.

- **stepper_category**
  
  `Stepper::stepper_category`

  A tag type characterizing the category of the stepper. This type must be convertible to `controlled_stepper_tag`.

**Notation**

- **ControlledStepper**
  A type that is a model of Controlled Stepper

- **State**
  A type representing the state \( x \) of the ODE

- **Time**
  A type representing the time \( t \) of the ODE

- **stepper**
  An object of type `ControlledStepper`

- **x**
  Object of type `State`

- **t, dt**
  Objects of type `Time`

- **sys**
  An object defining the ODE, should be a model of `System`, `Symplectic System`, `Simple Symplectic System` or `Implicit System`.

**Valid Expressions**

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do step</td>
<td><code>step_per_try_step(sys, x, t, dt)</code></td>
<td><code>controlled_step_result</code></td>
<td>Tries one step of step size ( dt ). If the step was successful, <code>success</code> is returned, the resulting state is written to ( x ), the new time is stored in ( t ) and ( dt ) now contains a new (possibly larger) step-size for the next step. If the error was too big, <code>rejected</code> is returned and the results are neglected - ( x ) and ( t ) are unchanged and ( dt ) now contains a reduced step-size to be used for the next try.</td>
</tr>
</tbody>
</table>
Models

• controlled_error_stepper< runge_kutta_cash_karp54 >
• controlled_error_stepper_fsal< runge_kutta_dopri5 >
• controlled_error_stepper< runge_kutta_fehlberg78 >
• rosenbrock4_controller
• bulirsch_stoer

Dense Output Stepper

This concept specifies the interface a dense output stepper has to fulfill to be used within integrate functions.

Description

A dense output stepper following this Dense Output Stepper concept provides the possibility to perform a single step of the solution \( x(t) \) of an ODE to obtain \( x(t+\Delta t) \). The step-size \( \Delta t \) might be adjusted automatically due to error control. Dense output steppers also can interpolate the solution to calculate the state \( x(t') \) at any point \( t \leq t' \leq t+\Delta t \).

Associated types

• state_type

  Stepper::state_type

  The type characterizing the state of the ODE, hence \( x \).

• deriv_type

  Stepper::deriv_type

  The type characterizing the derivative of the ODE, hence \( dx/dt \).

• time_type

  Stepper::time_type

  The type characterizing the dependent variable of the ODE, hence the time \( t \).

• value_type

  Stepper::value_type

  The numerical data type which is used within the stepper, something like float, double, complex\lt; double &gt;.

• stepper_category

  Stepper::stepper_category

  A tag type characterizing the category of the stepper. This type must be convertible to dense_output_stepper_tag.

Notation

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stepper</td>
<td>A type that is a model of Dense Output Stepper</td>
</tr>
<tr>
<td>State</td>
<td>A type representing the state ( x ) of the ODE</td>
</tr>
<tr>
<td>stepper</td>
<td>An object of type Stepper</td>
</tr>
</tbody>
</table>
x₀, x Object of type State

t₀, dt₀, t Objects of type Stepper::time_type

sys An object defining the ODE, should be a model of System, Symplectic System, Simple Symplectic System or Implicit System.

### Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initialize integration</td>
<td>stepper.initialize( x₀ , t₀ , dt₀ )</td>
<td>void</td>
<td>Initializes the stepper with initial values x₀, t₀ and dt₀.</td>
</tr>
<tr>
<td>Do step</td>
<td>stepper.do_step( sys )</td>
<td>std::pair&lt; Stepper::time_type, Stepper::time_type &gt;</td>
<td>Performs one step using the ODE defined by sys. The step-size might be changed internally due to error control. This function returns a pair containing t and t+dt representing the interval for which interpolation can be performed.</td>
</tr>
<tr>
<td>Do interpolation</td>
<td>stepper.calc_state( t_inter, x )</td>
<td>void</td>
<td>Performs the interpolation to calculate ( x(t_{\text{inter}}) ) where ( t \leq t_{\text{inter}} \leq t+dt ).</td>
</tr>
<tr>
<td>Get current time</td>
<td>stepper.current_time()</td>
<td>const Stepper::time_type &amp;</td>
<td>Returns the current time ( t+dt ) of the stepper, that is the end time of the last step and the starting time for the next call of do_step.</td>
</tr>
<tr>
<td>Get current state</td>
<td>stepper.current_state()</td>
<td>const Stepper::state_type &amp;</td>
<td>Returns the current state of the stepper, that is ( x(t+dt) ), the state at the time returned by stepper.current_time().</td>
</tr>
</tbody>
</table>

### Models

- dense_output_controlled_explicit_fsal< controlled_error_stepper_fsal< runge_kutta_dopri5 >
- bulirsch_stoer_dense_out
- rosenbrock4_dense_output

### State Algebra Operations

#### Note

The following does not apply to implicit steppers like implicit_euler or Rosenbrock 4 as there the state_type can not be changed from ublas::vector and no algebra/operations are used.
Description

The State, Algebra and Operations together define a concept describing how the mathematical vector operations required for the stepper algorithms are performed. The typical vector operation done within steppers is

\[ y = \sum \alpha_i x_i \]

The State represents the state variable of an ODE, usually denoted with \( x \). Algorithmically, the state is often realized as a `vector<double>` or `array<double, N>`, however, the genericity of odeint enables you to basically use anything as a state type. The algorithmic counterpart of such mathematical expressions is divided into two parts. First, the Algebra is used to account for the vector character of the equation. In the case of a vector as state type this means the Algebra is responsible for iteration over all vector elements. Second, the Operations are used to represent the actual operation applied to each of the vector elements. So the Algebra iterates over all elements of the States and calls an operation taken from the Operations for each element. This is where State, Algebra and Operations have to work together to make odeint running. Please have a look at the `range_algebra` and `default_operations` to see an example how this is implemented.

In the following we describe how State, Algebra and Operations are used together within the stepper implementations.

Operations

Notation

Operations

The operations type

Value1, ..., ValueN

Types representing the value or time type of stepper

Scale

Type of the scale operation

scale

Object of type Scale

ScaleSumN

Type that represents a general scale_sum operation, \( N \) should be replaced by a number from 1 to 14.

scale_sum

Object of type ScaleSumN, \( N \) should be replaced by a number from 1 to 14.

ScaleSumSwap2

Type of the scale sum swap operation

scale_sum_swap2

Object of type ScaleSumSwap2

a1, a2, ...

Objects of type Value1, Value2, ...

y, x1, x2, ...

Objects of State's value type
Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get scale operation</td>
<td>Operations::scale&lt; Value &gt;</td>
<td>Scale</td>
<td>Get Scale from Operations</td>
</tr>
<tr>
<td>Scale constructor</td>
<td>Scale&lt; Value &gt; ( a )</td>
<td>Scale</td>
<td>Constructs a Scale object</td>
</tr>
<tr>
<td>Scale operation</td>
<td>scale( x )</td>
<td>void</td>
<td>Calculates x *= a</td>
</tr>
<tr>
<td>Get general scale_sum operation</td>
<td>Operations::scale_sum&lt; Value1 , ... , ValueN &gt;</td>
<td>ScaleSumN</td>
<td>Get the ScaleSumN type from Operations, N should be replaced by a number from 1 to 14.</td>
</tr>
<tr>
<td>scale_sum constructor</td>
<td>ScaleSumN&lt; Value1 , ... , ValueN &gt; ( a1 , ... , aN )</td>
<td>ScaleSumN</td>
<td>Constructs a scale_sum object given N parameter values with N between 1 and 14.</td>
</tr>
<tr>
<td>scale_sum operation</td>
<td>scale_sumN( y , x1 , ... , xN )</td>
<td>void</td>
<td>Calculates y = a1<em>x1 + a2</em>x2 + ... + aN*xN. Note that this is an N+1-ary function call.</td>
</tr>
<tr>
<td>Get scale sum swap operation</td>
<td>Operations::scale_sum_swap2&lt; Value1 , Value2 &gt;</td>
<td>ScaleSumSwap2</td>
<td>Get scale sum swap from operations</td>
</tr>
<tr>
<td>ScaleSumSwap2 constructor</td>
<td>ScaleSumSwap2&lt; Value1 , Value2 &gt; ( a1 , a2 )</td>
<td>ScaleSumSwap2</td>
<td>Constructor</td>
</tr>
<tr>
<td>ScaleSumSwap2 operation</td>
<td>scale_sum_swap2( x1 , x2 , x3 )</td>
<td>void</td>
<td>Calculates tmp = x1, x1 = a1<em>x2 + a2</em>x3 and x2 = tmp.</td>
</tr>
</tbody>
</table>

Algebra

Notation

State  
The state type

Algebra  
The algebra type

OperationN  
An N-ary operation type, N should be a number from 1 to 14.

algebra  
Object of type Algebra

operationN  
Object of type OperationN

y, x1, x2, ...  
Objects of type State
**Valid Expressions**

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector Operation with arity 2</td>
<td><code>algebra.for_each2(y, x, operation2)</code></td>
<td><code>void</code></td>
<td>Calls <code>operation2(y_i, x_i)</code> for each element <code>y_i</code> of <code>y</code> and <code>x_i</code> of <code>x</code>.</td>
</tr>
<tr>
<td>Vector Operation with arity 3</td>
<td><code>algebra.for_each3(y, x1, x2, operation3)</code></td>
<td><code>void</code></td>
<td>Calls <code>operation3(y_i, x1_i, x2_i)</code> for each element <code>y_i</code> of <code>y</code> and <code>x1_i</code> of <code>x1</code> and <code>x2_i</code> of <code>x2</code>.</td>
</tr>
<tr>
<td>Vector Operation with arity N</td>
<td><code>algebra.for_eachN(y, x1, ..., xN, operationN)</code></td>
<td><code>void</code></td>
<td>Calls <code>operationN(y_i, x1_i, ..., xN_i)</code> for each element <code>y_i</code> of <code>y</code> and <code>x1_i</code> of <code>x1</code> and so on. <code>N</code> should be replaced by a number between 1 and 14.</td>
</tr>
</tbody>
</table>

**Pre-Defined implementations**

As standard configuration odeint uses the `range_algebra` and `default_operations` which suffices most situations. However, a few more possibilities exist either to gain better performance or to ensure interoperability with other libraries. In the following we list the existing `Algebra/Operations` configurations that can be used in the steppers.

<table>
<thead>
<tr>
<th>State</th>
<th>Algebra</th>
<th>Operations</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anything supporting <code>Boost.Range</code>, like <code>std::vector</code>, <code>std::list</code>, <code>boost::array</code>,... based on a <code>value_type</code> that supports operators <code>+,*</code> (typically <code>double</code>)</td>
<td><code>range_algebra</code></td>
<td><code>default_operations</code></td>
<td>Standard implementation, applicable for most typical situations.</td>
</tr>
<tr>
<td><code>boost::array</code> based on a <code>value_type</code> that supports operators <code>+,*</code></td>
<td><code>array_algebra</code></td>
<td><code>default_operations</code></td>
<td>Special implementation for <code>boost::array</code> with better performance than <code>range_algebra</code></td>
</tr>
<tr>
<td>Anything that defines operators <code>+</code> within itself and <code>*</code> with <code>scalar</code> (Mathematically spoken, anything that is a vector space).</td>
<td><code>vector_space_algebra</code></td>
<td><code>default_operations</code></td>
<td>For the use of Controlled Stepper, the template <code>vector_space_reduce</code> has to be instantiated.</td>
</tr>
<tr>
<td><code>thrust::device_vector</code>, <code>thrust::host_vector</code></td>
<td><code>thrust_algebra</code></td>
<td><code>thrust_operations</code></td>
<td>For running odeint on CUDA devices by using Thrust</td>
</tr>
<tr>
<td><code>boost::array</code> or anything which allocates the elements in a C-like manner</td>
<td><code>vector_space_algebra</code></td>
<td><code>mkl_operations</code></td>
<td>Using the Intel Math Kernel Library in odeint for maximum performance. Currently, only the RK4 stepper is supported.</td>
</tr>
</tbody>
</table>
## Example expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector operation</td>
<td>algebra.for_each3( y, x1, x2, Operations::scale_sum2&lt; Value1, Value2 &gt;( a1, a2 ))</td>
<td>void</td>
<td>Calculates $y = a1 \times x1 + a2 \times x2$</td>
</tr>
</tbody>
</table>

## State Wrapper

### Description

The State Wrapper concept describes the way odeint creates temporary state objects to store intermediate results within the stepper's `do_step` methods.

### Notation

- **State**: A type that is the `state_type` of the ODE
- **WrappedState**: A type that is a model of State Wrapper for the state type `State`.
- **x**: Object of type `State`
- **w**: Object of type `WrappedState`

### Valid Expressions

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Type</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Get resizeability</td>
<td><code>is_resizeable&lt; State &gt;</code></td>
<td><code>boost::false_type</code> or <code>boost::true_type</code></td>
<td>Returns <code>boost::true_type</code> if the <code>State</code> is resizeable, otherwise.</td>
</tr>
<tr>
<td>Create WrappedState type</td>
<td><code>state_wrapper&lt; State &gt;</code></td>
<td><code>WrappedState</code></td>
<td>Creates the type for a WrappedState for the state type <code>State</code></td>
</tr>
<tr>
<td>Constructor</td>
<td><code>WrappedState()</code></td>
<td><code>WrappedState</code></td>
<td>Constructs a state wrapper with an empty state</td>
</tr>
<tr>
<td>Copy Constructor</td>
<td><code>WrappedState( w )</code></td>
<td><code>WrappedState</code></td>
<td>Constructs a state wrapper with a state of the same size as the state in <code>w</code></td>
</tr>
<tr>
<td>Get state</td>
<td><code>w.m_v</code></td>
<td><code>State</code></td>
<td>Returns the <code>State</code> object of this state wrapper.</td>
</tr>
</tbody>
</table>
**Literature**

**General information about numerical integration of ordinary differential equations:**


**Symplectic integration of numerical integration:**


**Special symplectic methods:**


**Special systems:**

[8] Fermi-Pasta-Ulam nonlinear lattice oscillations

Acknowledgments

• Steven Watanabe for managing the Boost review process.
• All people who participated in the odeint review process on the Boost mailing list.
• Paul Bristow for helping with the documentation.
• The Google Summer Of Code (GSOC) program for funding and Andrew Sutton for supervising us during the GSOC and for lots of useful discussions and feedback about many implementation details.
• Joachim Faulhaber for motivating us to participate in the Boost review process and many detailed comments about the library.
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Contributers

• Andreas Angelopoulos implemented the sparse matrix implicit Euler stepper using the MTL4 library.
• Rajeev Singh implemented the stiff Van der Pol oscillator example.
• Sylwester Arabas improved the documentation.
• Denis Demidov provided the adaption to the VexCL and Viennacl libraries.
• Christoph Koke provided improved binders.
• Lee Hodgkinson provided the black hole example.
• Michael Morin fixed several typos in the documentation and the source code comments.
odeint Reference

Header `<boost/numeric/odeint/integrate/integrate.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename System, typename State, typename Time,
                    typename Observer>
            size_t integrate(System, State &, Time, Time, Time, Observer);
        }
    }
}
```

Function template integrate

boost::numeric::odeint::integrate — Integrates the ODE.

Synopsis

```
// In header: <boost/numeric/odeint/integrate/integrate.hpp>

template<typename System, typename State, typename Time, typename Observer>
size_t integrate(System system, State & start_state, Time start_time,
                  Time end_time, Time dt, Observer observer);
```

Description

Integrates the ODE given by system from start_time to end_time starting with start_state as initial condition and dt as initial time step. This function uses a dense output dopri5 stepper and performs an adaptive integration with step size control, thus dt changes during the integration. This method uses standard error bounds of 1E-6. After each step, the observer is called.

Parameters:
- `dt` Initial step size, will be adjusted during the integration.
- `end_time` End time of the integration.
- `observer` Observer that will be called after each time step.
- `start_state` The initial state.
- `start_time` Start time of the integration.
- `system` The system function to solve, hence the r.h.s. of the ordinary differential equation.

Returns: The number of steps performed.

Function template integrate

boost::numeric::odeint::integrate — Integrates the ODE without observer calls.
Synopsis

```cpp
template<typename System, typename State, typename Time>
size_t integrate(System system, State & start_state, Time start_time,
                 Time end_time, Time dt);
```

Description

Integrates the ODE given by system from start_time to end_time starting with start_state as initial condition and dt as initial time step. This function uses a dense output dopri5 stepper and performs an adaptive integration with step size control, thus dt changes during the integration. This method uses standard error bounds of 1E-6. No observer is called.

Parameters:
- **dt**: Initial step size, will be adjusted during the integration.
- **end_time**: End time of the integration.
- **start_state**: The initial state.
- **start_time**: Start time of the integration.
- **system**: The system function to solve, hence the r.h.s. of the ordinary differential equation.

Returns: The number of steps performed.

Header `<boost/numeric/odeint/integrate/integrate_adaptive.hpp>`

```cpp
namespace boost {
  namespace numeric {
    namespace odeint {
      template<typename Stepper, typename System, typename State,
               typename Time, typename Observer>
      size_t integrate_adaptive(Stepper, System, State &, Time, Time, Time,
                                  Observer);

      // Second version to solve the forwarding problem, can be called with Boost.Range as start_state.
      template<typename Stepper, typename System, typename State,
               typename Time, typename Observer>
      size_t integrate_adaptive(Stepper stepper, System system,
                                  const State & start_state, Time start_time,
                                  Time end_time, Time dt, Observer observer);

      // integrate_adaptive without an observer.
      template<typename Stepper, typename System, typename State,
               typename Time>
      size_t integrate_adaptive(Stepper stepper, System system,
                                  State & start_state, Time start_time,
                                  Time end_time, Time dt);

      // Second version to solve the forwarding problem, can be called with Boost.Range as start_state.
      template<typename Stepper, typename System, typename State,
               typename Time>
      size_t integrate_adaptive(Stepper stepper, System system,
                                  const State & start_state, Time start_time,
                                  Time end_time, Time dt);
    }
  }
}
```
**Function template integrate_adaptive**

boost::numeric::odeint::integrate_adaptive — Integrates the ODE with adaptive step size.

**Synopsis**

```cpp
// In header: <boost/numeric/odeint/integrate/integrate_adaptive.hpp>

template<typename Stepper, typename System, typename State, typename Time,
         typename Observer>
size_t integrate_adaptive(Stepper stepper, System system,
                         State & start_state, Time start_time,
                         Time end_time, Time dt, Observer observer);
```

**Description**

This function integrates the ODE given by system with the given stepper. The observer is called after each step. If the stepper has no error control, the step size remains constant and the observer is called at equidistant time points t0+n*dt. If the stepper is a ControlledStepper, the step size is adjusted and the observer is called in non-equidistant intervals.

**Parameters:**
- `dt` The time step between observer calls, *not* necessarily the time step of the integration.
- `end_time` The final integration time tend.
- `observer` Function/Functor called at equidistant time intervals.
- `start_state` The initial condition x0.
- `start_time` The initial time t0.
- `stepper` The stepper to be used for numerical integration.
- `system` Function/Functor defining the rhs of the ODE.

**Returns:** The number of steps performed.
Function template integrate_const

boost::numeric::odeint::integrate_const — Integrates the ODE with constant step size.

Synopsis

// In header: <boost/numeric/odeint/integrate/integrate_const.hpp>

template<typename Stepper, typename System, typename State, typename Time, 
typename Observer>
size_t integrate_const(Stepper stepper, System system, 
const State & start_state, Time start_time, 
Time end_time, Time dt, Observer observer);

Description

Integrates the ODE defined by system using the given stepper. This method ensures that the observer is called at constant intervals dt. If the Stepper is a normal stepper without step size control, dt is also used for the numerical scheme. If a ControlledStepper is provided, the algorithm might reduce the step size to meet the error bounds, but it is ensured that the observer is always called at equidistant time points $t_0 + n \times dt$. If a DenseOutputStepper is used, the step size also may vary and the dense output is used to call the observer at equidistant time points.
Parameters:

dt The time step between observer calls, not necessarily the time step of the integration.
end_time The final integration time tend.
observer Function/Functor called at equidistant time intervals.
start_state The initial condition x0.
start_time The initial time t0.
stepper The stepper to be used for numerical integration.
system Function/Functor defining the rhs of the ODE.

Returns: The number of steps performed.

Header `<boost/numeric/odeint/integrate/integrate_n_steps.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper, typename System, typename State, typename Time, typename Observer>
            Time integrate_n_steps(Stepper, System, State &, Time, Time, size_t, Observer);

            // Solves the forwarding problem, can be called with Boost.Range as start_state.
            template<typename Stepper, typename System, typename State, typename Time, typename Observer>
            Time integrate_n_steps(Stepper stepper, System system,
                                    State & start_state, Time start_time, Time dt, size_t num_of_steps,
                                    Observer observer);

            // The same function as above, but without observer calls.
            template<typename Stepper, typename System, typename State, typename Time>
            Time integrate_n_steps(Stepper stepper, System system,
                                    State & start_state, Time start_time, Time dt, size_t num_of_steps);

            // Solves the forwarding problem, can be called with Boost.Range as start_state.
            template<typename Stepper, typename System, typename State, typename Time>
            Time integrate_n_steps(Stepper stepper, System system,
                                    State & start_state, Time start_time, Time dt, size_t num_of_steps);
        }
    }
}
```

Function template integrate_n_steps

`boost::numeric::odeint::integrate_n_steps` — Integrates the ODE with constant step size.

Synopsis

```cpp
// In header: `<boost/numeric/odeint/integrate/integrate_n_steps.hpp>`

template<typename Stepper, typename System, typename State, typename Time, typename Observer>
    Time integrate_n_steps(Stepper stepper, System system, State & start_state,
                            Time start_time, Time dt, size_t num_of_steps,
                            Observer observer);
```
Description

This function is similar to integrate_const. The observer is called at equidistant time intervals \( t_0 + n \cdot dt \). If the Stepper is a normal stepper without step size control, \( dt \) is also used for the numerical scheme. If a ControlledStepper is provided, the algorithm might reduce the step size to meet the error bounds, but it is ensured that the observer is always called at equidistant time points \( t_0 + n \cdot dt \). If a DenseOutputStepper is used, the step size also may vary and the dense output is used to call the observer at equidistant time points. The final integration time is always \( t_0 + \text{num_of_steps} \cdot dt \).

Parameters:
- \( dt \): The time step between observer calls, not necessarily the time step of the integration.
- \( \text{num_of_steps} \): Number of steps to be performed
- \( \text{observer} \): Function/Functor called at equidistant time intervals.
- \( \text{start_state} \): The initial condition \( x_0 \).
- \( \text{start_time} \): The initial time \( t_0 \).
- \( \text{stepper} \): The stepper to be used for numerical integration.
- \( \text{system} \): Function/Functor defining the rhs of the ODE.

Returns: The number of steps performed.

Header `<boost/numeric/odeint/integrate/integrate_times.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper, typename System, typename State,
                     typename TimeIterator, typename Time, typename Observer>
            size_t integrate_times(Stepper, System, State &, TimeIterator,
                                   TimeIterator, Time, Observer);

            // Solves the forwarding problem, can be called with Boost.Range as start_state.
            template<typename Stepper, typename System, typename State,
                     typename TimeIterator, typename Time, typename Observer>
            size_t integrate_times(Stepper stepper, System system,
                                   const State & start_state,
                                   TimeIterator times_start,
                                   TimeIterator times_end, Time dt,
                                   Observer observer);

            // The same function as above, but without observer calls.
            template<typename Stepper, typename System, typename State,
                     typename TimeRange, typename Time, typename Observer>
            size_t integrate_times(Stepper stepper, System system,
                                   const State & start_state,
                                   const TimeRange & times,
                                   Time dt, Observer observer);

            // Solves the forwarding problem, can be called with Boost.Range as start_state.
            template<typename Stepper, typename System, typename State,
                     typename TimeRange, typename Time, typename Observer>
            size_t integrate_times(Stepper stepper, System system,
                                   const State & start_state,
                                   const TimeRange & times, Time dt,
                                   Observer observer);
        }
    }
}
```

Function template `integrate_times`

`boost::numeric::odeint::integrate_times` — Integrates the ODE with observer calls at given time points.
**Synopsis**

```cpp
// In header: <boost/numeric/odeint/integrate/integrate_times.hpp>

template<typename Stepper, typename System, typename State, typename TimeIterator, typename Time, typename Observer>
size_t integrate_times(Stepper stepper, System system, State & start_state, TimeIterator times_start, TimeIterator times_end, Time dt, Observer observer);
```

**Description**

Integrates the ODE given by system using the given stepper. This function does observer calls at the subsequent time points given by the range times_start, times_end. If the stepper has not step size control, the step size might be reduced occasionally to ensure observer calls exactly at the time points from the given sequence. If the stepper is a ControlledStepper, the step size is adjusted to meet the error bounds, but also might be reduced occasionally to ensure correct observer calls. If a DenseOutputStepper is provided, the dense output functionality is used to call the observer at the given times. The end time of the integration is always *(end_time-1)*.

**Parameters:**
- `dt` The time step between observer calls, *not* necessarily the time step of the integration.
- `observer` Function/Functor called at equidistant time intervals.
- `start_state` The initial condition x0.
- `stepper` The stepper to be used for numerical integration.
- `system` Function/Functor defining the rhs of the ODE.
- `times_end` Iterator to the end time
- `times_start` Iterator to the start time

**Returns:** The number of steps performed.

---

**Header `<boost/numeric/odeint/integrate/null_observer.hpp>`**

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            struct null_observer;
        }
    }
}
```

**Struct null_observer**

`boost::numeric::odeint::null_observer`

**Synopsis**

```cpp
// In header: <boost/numeric/odeint/integrate/null_observer.hpp>

struct null_observer {

    // public member functions
    template<typename State, typename Time>
    void operator()(const State &, Time) const;
};
```
null_observer public member functions

1. template<typename State, typename Time>
   void operator()(const State &, Time) const;

Header `<boost/numeric/odeint/integrate/observer_collection.hpp>`

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Time> class observer_collection;
        }
    }
}

Class template observer_collection

boost::numeric::odeint::observer_collection

Synopsis

// In header: `<boost/numeric/odeint/integrate/observer_collection.hpp>`

template<typename State, typename Time>
class observer_collection {
public:
    // types
    typedef boost::function< void(const State &, const Time &) > observer_type;
    typedef std::vector< observer_type > collection_type;

    // public member functions
    void operator()(const State &, Time);
    collection_type & observers(void);
    const collection_type & observers(void) const;
};

Description

observer_collection public member functions

1. void operator()(const State & x, Time t);

2. collection_type & observers(void);

3. const collection_type & observers(void) const;
Header `<boost/numeric/odeint/stepper/adams_bashforth.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<size_t Steps, typename State, typename Value = double, 
                    typename Deriv = State, typename Time = Value, 
                    typename Algebra = range_algebra, 
                    typename Operations = default_operations, 
                    typename Resizer = initially_resizer, 
                    typename InitializingStepper = runge_kutta4< State , Value , Deriv , Time , Algebra , Operations , Resizer > >
            class adams_bashforth;
        }
    }
}
```

Class template `adams_bashforth`

`boost::numeric::odeint::adams_bashforth` — The Adams-Bashforth multistep algorithm.
Synopsis

// In header: <boost/numeric/odeint/stepper/adams_bashforth.hpp>

template<size_t Steps, typename State, typename Value = double,
    typename Deriv = State, typename Time = Value,
    typename Algebra = range_algebra,
    typename Operations = default_operations,
    typename Resizer = initially_resizer,
    typename InitializingStepper = runge_kutta4< State , Value , Deriv , Time , Algebra , Operations, Resizer > >
class adams_bashforth :
    public boost::numeric::odeint::algebra_stepper_base< Algebra, Operations >
{
public:
    // types
    typedef State state_type;
    typedef state_wrapper< state_type > wrapped_state_type;
    typedef Value value_type;
    typedef Deriv deriv_type;
    typedef state_wrapper< deriv_type > wrapped_deriv_type;
    typedef Time time_type;
    typedef Resizer resizer_type;
    typedef stepper_tag stepper_category;
    typedef InitializingStepper initializing_stepper_type;
    typedef algebra_stepper_base< Algebra, Operations >::algebra_type algebra_type;
    typedef algebra_stepper_base< Algebra, Operations >::operations_type operations_type;
    typedef unsigned short order_type;
    typedef unspecified step_storage_type;

    // construct/copy/destruct
    adams_bashforth(const algebra_type & = algebra_type());
    adams_bashforth(const adams_bashforth &);
    adams_bashforth& operator=(const adams_bashforth &);

    // public member functions
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename StateIn>
    void do_step(System, const StateIn &, time_type, StateOut &, time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, const StateOut &, time_type);
    template<typename StateType> void adjust_size(const StateType &);
    const step_storage_type & step_storage(void) const;
    step_storage_type & step_storage(void);
    template<typename ExplicitStepper, typename System, typename StateIn>
Description

The Adams-Bashforth method is a multi-step algorithm with configurable step number. The step number is specified as template parameter Steps and it then uses the result from the previous Steps steps. See also en.wikipedia.org/wiki/Linear_multistep_method. Currently, a maximum of Steps=8 is supported. The method is explicit and fulfills the Stepper concept. Step size control or continuous output are not provided.

This class derives from algebra_base and inherits its interface via CRTP (current recurring template pattern). For more details see algebra_stepper_base.

Template Parameters

1. size_t Steps
   The number of steps (maximal 8).

2. typename State
   The state type.

3. typename Value = double
   The value type.

4. typename Deriv = State
   The type representing the time derivative of the state.

5. typename Time = Value
   The time representing the independent variable - the time.

6. typename Algebra = range_algebra
The algebra type.

7. `typename Operations = default_operations`

The operations type.

8. `typename Resizer = initially_resizer`

The resizer policy type.

9. `typename InitializingStepper = runge_kutta4< State , Value , Deriv , Time , Algebra , Operations , Resizer >`

The stepper for the first two steps.

adams_bashforth public construct/copy/destroy

1. `adams_bashforth(const algebra_type & algebra = algebra_type());`

   Constructs the `adams_bashforth` class. This constructor can be used as a default constructor if the algebra has a default constructor.

   Parameters: `algebra` A copy of algebra is made and stored.

2. `adams_bashforth(const adams_bashforth & stepper);`

3. `adams_bashforth& operator=(const adams_bashforth & stepper);`

adams_bashforth public member functions

1. `order_type order(void) const;`

   Returns the order of the algorithm, which is equal to the number of steps.

   Returns: `order` of the method.

2. `template<typename System, typename StateInOut> void do_step(System system, StateInOut & x, time_type t, time_type dt);`

   This method performs one step. It transforms the result in-place.

   Parameters: `dt` The step size.
   `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
   `t` The value of the time, at which the step should be performed.
   `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in `x`.

3. `template<typename System, typename StateInOut> void do_step(System system, const StateInOut & x, time_type t, time_type dt);`

   Second version to solve the forwarding problem, can be called with Boost.Range as `StateInOut`. 

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4. template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out, 
                              time_type dt);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place.

Parameters:
- **dt**  The step size.
- **in**  The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.

5. template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, 
                                const StateOut & out, time_type dt);

Second version to solve the forwarding problem, can be called with Boost.Range as StateOut.

6. template<typename StateType> void adjust_size(const StateType & x);

Adjust the size of all temporaries in the stepper manually.

Parameters:  
- **x**  A state from which the size of the temporaries to be resized is deduced.

7. const step_storage_type & step_storage(void) const;

Returns the storage of intermediate results.

Returns:  
- The storage of intermediate results.

8. step_storage_type & step_storage(void);

Returns the storage of intermediate results.

Returns:  
- The storage of intermediate results.

9. template<typename ExplicitStepper, typename System, typename StateIn>
   void initialize(ExplicitStepper explicit_stepper, System system, 
                                 StateIn & x, time_type & t, time_type dt);

Initialized the stepper. Does Steps-1 steps with the explicit_stepper to fill the buffer.

Parameters:
- **dt**  The step size.
- **explicit_stepper**  the stepper used to fill the buffer of previous step results.
- **system**  The system function to solve, hence the r.h.s. of the ordinary differential equation. 
  It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.
- **x**  The state of the ODE which should be solved. After calling do_step the result is updated in x.

10. template<typename System, typename StateIn>
    void initialize(System system, StateIn & x, time_type & t, time_type dt);

Initialized the stepper. Does Steps-1 steps with an internal instance of InitializingStepper to fill the buffer.
The state \( x \) and time \( t \) are updated to the values after Steps-1 initial steps.

**Parameters:**
- \( dt \) The step size.
- \( system \) The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- \( t \) The initial value of the time, updated in this method.
- \( x \) The initial state of the ODE which should be solved, updated in this method.

**Private Member Functions**

1. ```cpp
   template<typename System, typename StateIn, typename StateOut>
   void do_step_impl(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);
```

2. ```cpp
   template<typename StateIn>
   bool resize_impl(const StateIn & x);
```
Header `<boost/numeric/odeint/stepper/adams_bashforth_moulton.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<size_t Steps, typename State, typename Value = double,
                     typename Deriv = State, typename Time = Value,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
            class adams_bashforth_moulton;
        }
    }
}
```

Class template adams_bashforth_moulton

`boost::numeric::odeint::adams_bashforth_moulton` — The Adams-Bashforth-Moulton multistep algorithm.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/adams_bashforth_moulton.hpp>

template<size_t Steps, typename State, typename Value = double,
         typename Deriv = State, typename Time = Value,
         typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class adams_bashforth_moulton {
public:
    // types
    typedef State state_type;
    typedef state_wrapper<state_type> wrapped_state_type;
    typedef Value value_type;
    typedef Deriv deriv_type;
    typedef state_wrapper<deriv_type> wrapped_deriv_type;
    typedef Time time_type;
    typedef Algebra algebra_type;
    typedef Operations operations_type;
    typedef Resizer resizer_type;
    typedef stepper_tag stepper_category;
    typedef unsigned short order_type;
    // construct/copy/destroct
    adams_bashforth_moulton(void);
    adams_bashforth_moulton(const algebra_type &);
    // public member functions
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &time_type, time_type);
    template<typename System, typename StateInOut>
    void do_step(System, const StateInOut &, time_type, time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, const StateOut &, time_type);
    template<typename System, typename StateIn>
    void initialize(System, StateIn &, time_type &);
    // public data members
    static const size_t steps;
    static const order_type order_value;
};
```

Description

The Adams-Bashforth method is a multi-step predictor-corrector algorithm with configurable step number. The step number is specified as template parameter Steps and it then uses the result from the previous Steps steps. See also en.wikipedia.org/wiki/Linear_multistep_method. Currently, a maximum of Steps=8 is supported. The method is explicit and fulfills the Stepper concept. Step size control or continuous output are not provided.

This class derives from algebra_base and inherits its interface via CRTP (current recurring template pattern). For more details see algebra_stepper_base.
Template Parameters

1. `size_t Steps`
   The number of steps (maximal 8).

2. `typename State`
   The state type.

3. `typename Value = double`
   The value type.

4. `typename Deriv = State`
   The type representing the time derivative of the state.

5. `typename Time = Value`
   The time representing the independent variable - the time.

6. `typename Algebra = range_algebra`
   The algebra type.

7. `typename Operations = default_operations`
   The operations type.

8. `typename Resizer = initially_resizer`
   The resizer policy type.

**adams_bashforth_moulton public construct/copy/destroy**

1. `adams_bashforth_moulton(void);`
   Constructs the `adams_bashforth` class.

2. `adams_bashforth_moulton(const algebra_type & algebra);`
   Constructs the `adams_bashforth` class. This constructor can be used as a default constructor if the algebra has a default constructor.
   Parameters: `algebra` A copy of algebra is made and stored.

**adams_bashforth_moulton public member functions**

1. `order_type order(void) const;`
   Returns the order of the algorithm, which is equal to the number of steps+1.
Returns: order of the method.

2. ```template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);
```

This method performs one step. It transforms the result in-place.

Parameters:
- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.

3. ```template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

4. ```template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, const StateOut & out, time_type dt);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place.

Parameters:
- **dt**: The step size.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

5. ```template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateOut.

6. ```template<typename StateType> void adjust_size(const StateType & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters:
- **x**: A state from which the size of the temporaries to be resized is deduced.

7. ```template<typename ExplicitStepper, typename System, typename StateIn>
   void initialize(ExplicitStepper explicit_stepper, System system, StateIn & x, time_type & t, time_type dt);
```

Initialized the stepper. Does Steps-1 steps with the explicit_stepper to fill the buffer.

**Note**

The state x and time t are updated to the values after Steps-1 initial steps.

Parameters:
- **dt**: The step size.
- **explicit_stepper**: The stepper used to fill the buffer of previous step results.
The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept. The initial state of the ODE which should be solved, updated after in this method.

### Header `<boost/numeric/odeint/stepper/adams_moulton.hpp>`

```cpp
template<typename System, typename StateIn>
void initialize(System system, StateIn & x, time_type & t, time_type dt);
```

Initialized the stepper. Does Steps-1 steps using the standard initializing stepper of the underlying `adams_bashforth` stepper.

**Parameters:**
- `dt`: The step size.
- `system`: The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- `t`: The value of the time, at which the step should be performed.
- `x`: The state of the ODE which should be solved. After calling `do_step` the result is updated in `x`.

**Class template `adams_moulton`**

```cpp
boost::numeric::odeint::adams_moulton
```
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/adams_moulton.hpp>

template<size_t Steps, typename State, typename Value = double,
           typename Deriv = State, typename Time = Value,
           typename Algebra = range_algebra,
           typename Operations = default_operations,
           typename Resizer = initially_resizer>

class adams_moulton {
public:
    // types
    typedef State state_type;
    typedef state_wrapper< state_type > wrapped_state_type;
    typedef Value value_type;
    typedef Deriv deriv_type;
    typedef deriv_type wrapped_deriv_type;
    typedef Time time_type;
    typedef Algebra algebra_type;
    typedef Operations operations_type;
    typedef Resizer resizer_type;
    typedef stepper_tag step_per_category;
    typedef adams_moulton< Steps, State, Value, Deriv, Time, Algebra, Operations, Resizer > step_per_type;
    typedef unsigned short order_type;
    typedef unspecified step_storage_type;

    // construct/copy/destruct
    adams_moulton();
    adams_moulton(algebra_type &);
    adams_moulton & operator=(const adams_moulton &);

    // public member functions
    order_type order(void) const;
    template<typename System, typename StateInOut, typename ABBuf>
    void do_step(System, StateInOut &, time_type, time_type, const ABBuf &);
    template<typename System, typename StateInOut, typename ABBuf>
    void do_step(System, const StateInOut &, time_type, time_type,
                 const ABBuf &);
    template<typename System, typename StateIn, typename StateOut,
             typename ABBuf>
    void do_step(System, const StateIn &, time_type, StateOut &, time_type,
                 const ABBuf &);
    template<typename StateType> void adjust_size(const StateType &);

    algebra_type & algebra();
    const algebra_type & algebra() const;

};
```
// private member functions
template<typename StateIn> bool resize_impl(const StateIn &);

// public data members
static const size_t steps;
static const order_type order_value;
};

Description

adams_moulton public construct/copy/destruct

1. adams_moulton();

2. adams_moulton(algebra_type & algebra);

3. adams_moulton & operator=(const adams_moulton & stepper);

adams_moulton public member functions

1. order_type order(void) const;

2. template<typename System, typename StateInOut, typename ABBuf>
   void do_step(System system, StateInOut & in, time_type t, time_type dt,
   const ABBuf & buf);

3. template<typename System, typename StateInOut, typename ABBuf>
   void do_step(System system, const StateInOut & in, time_type t,
   time_type dt, const ABBuf & buf);

4. template<typename System, typename StateIn, typename StateOut, typename ABBuf>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out,
   time_type dt, const ABBuf & buf);

5. template<typename System, typename StateIn, typename StateOut, typename ABBuf>
   void do_step(System system, const StateIn & in, time_type t,
   const StateOut & out, time_type dt, const ABBuf & buf);

6. template<typename StateType> void adjust_size(const StateType & x);

7. algebra_type & algebra();
**const algebra_type & algebra() const;**

**adams_moulton** private member functions

1. `template<typename StateIn> bool resize_impl(const StateIn & x);`

**Header `<boost/numeric/odeint/stepper/base/algebra_stepper_base.hpp>`**

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Algebra, typename Operations> class algebra_stepper_base;
        }
    }
}
```

**Class template algebra_stepper_base**

boost::numeric::odeint::algebra_stepper_base — Base class for all steppers with algebra and operations.

**Synopsis**

```cpp
// In header: `<boost/numeric/odeint/stepper/base/algebra_stepper_base.hpp>

template<typename Algebra, typename Operations>
class algebra_stepper_base {
    public:
        // types
        typedef Algebra algebra_type;
        typedef Operations operations_type;

        // construct/copy/destroy
        algebra_stepper_base(const algebra_type & algebra_type);
        algebra_stepper_base();
        const algebra_type & algebra();
        const algebra_type & algebra() const;
    }
```

**Description**

This class serves as a base class for all steppers with algebra and operations. It holds the algebra and provides access to the algebra. The operations are not instantiated, since they are static classes inside the operations class.

**Template Parameters**

1. **typename Algebra**

   The type of the algebra. Must fulfill the Algebra Concept, at least partially to work with the stepper.

2. **typename Operations**
The type of the operations. Must fulfill the Operations Concept, at least partially to work with the stepper.

**algebra_stepper_base public construct/copy/destroy**

1. `algebra_stepper_base(const algebra_type & algebra = algebra_type());`

Constructs a `algebra_stepper_base` and creates the algebra. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters: `algebra` The `algebra_stepper_base` stores and uses a copy of algebra.

**algebra_stepper_base public member functions**

1. `algebra_type & algebra();`

Returns: A reference to the algebra which is held by this class.

2. `const algebra_type & algebra() const;`

Returns: A const reference to the algebra which is held by this class.

**Header `<boost/numeric/odeint/stepper/base/explicit_error_stepper_base.hpp>`**

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper, unsigned short Order, 
                unsigned short StepperOrder, unsigned short ErrorOrder, 
                typename State, typename Value, typename Deriv, typename Time, 
                typename Algebra, typename Operations, typename Resizer>
            class explicit_error_stepper_base;
        }
    }
}
```

**Class template explicit_error_stepper_base**

`boost::numeric::odeint::explicit_error_stepper_base` — Base class for explicit steppers with error estimation. This class can used with controlled steppers for step size control.
template<typename Stepper, unsigned short Order, unsigned short StepperOrder,
    unsigned short ErrorOrder, typename State, typename Value,
    typename Deriv, typename Time, typename Algebra, typename Operations,
    typename Resizer>
class explicit_error_stepper_base :
public boost::numeric::odeint::algebra_stepper_base< Algebra, Operations >
{
public:
// types
typedef algebra_stepper_base< Algebra, Operations > algebra_stepper_base_type;
typedef algebra_stepper_base_type::algebra_type algebra_type;
typedef State state_type;
typedef Value value_type;
typedef Deriv deriv_type;
typedef Time time_type;
typedef Resizer resizer_type;
typedef Stepper stepper_type;
typedef explicit_error_stepper_tag stepper_category;
typedef unsigned short order_type;

// construct/copy/destruct
explicit_error_stepper_base(const algebra_type & = algebra_type());

// public member functions
order_type order(void) const;
order_type stepper_order(void) const;
order_type error_order(void) const;
template<typename System, typename StateInOut>
do_step(System, StateInOut &, time_type, time_type);
template<typename System, typename StateInOut>
do_step(System, const StateInOut &, time_type, time_type);
template<typename System, typename StateInOut, typename DerivIn>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System, StateInOut &, const DerivIn &, time_type, time_type);
template<typename System, typename StateIn, typename StateOut>
do_step(System, const StateIn &, time_type, StateOut &, time_type,
    time_type);
template<typename System, typename StateInOut, typename Err>
do_step(System, StateInOut &, time_type, time_type, Err &);
template<typename System, typename StateInOut, typename Err>
do_step(System, const StateInOut &, time_type, time_type, Err &);
template<typename System, typename StateIn, typename StateOut, typename Err>
do_step(System, const StateIn &, time_type, StateOut &, time_type,
    time_type, Err &);

// In header: <boost/numeric/odeint/stepper/base/explicit_error_stepper_base.hpp>
template<typename StateIn> void adjust_size(const StateIn &);
algebra_type & algebra();
const algebra_type & algebra() const;

// private member functions
template<typename System, typename StateInOut>
void do_step_v1(System, StateInOut &, time_type, time_type);
template<typename System, typename StateInOut, typename Err>
void do_step_v5(System, StateInOut &, time_type, time_type, Err &);
template<typename StateIn> bool resize_impl(const StateIn &);
stepper_type & stepper(void);
const stepper_type & stepper(void) const;

// public data members
static const order_type order_value;
static const order_type stepper_order_value;
static const order_type error_order_value;

Description

This class serves as the base class for all explicit steppers with algebra and operations. In contrast to explicit_stepper_base it also estimates the error and can be used in a controlled stepper to provide step size control.

Note

This stepper provides do_step methods with and without error estimation. It has therefore three orders, one for the order of a step if the error is not estimated. The other two orders are the orders of the step and the error step if the error estimation is performed.

explicit_error_stepper_base is used as the interface in a CRTP (currently recurring template pattern). In order to work correctly the parent class needs to have a method do_step_impl( system , in , dxdt_in , t , out , dt , xerr ). explicit_error_stepper_base derives from algebra_stepper_base.

explicit_error_stepper_base provides several overloaded do_step methods, see the list below. Only two of them are needed to fulfill the Error Stepper concept. The other ones are for convenience and for performance. Some of them simply update the state out-of-place, while other expect that the first derivative at t is passed to the stepper.

- do_step( sys , x , t , dt ) - The classical do_step method needed to fulfill the Error Stepper concept. The state is updated in-place. A type modelling a Boost.Range can be used for x.
- do_step( sys , x , dxdt , t , dt ) - This method updates the state in-place, but the derivative at the point t must be explicitly passed in dxdt.
- do_step( sys , in , t , out , dt ) - This method updates the state out-of-place, hence the result of the step is stored in out.
- do_step( sys , in , dxdt , t , out , dt ) - This method update the state out-of-place and expects that the derivative at the point t is explicitly passed in dxdt. It is a combination of the two do_step methods above.
- do_step( sys , x , t , dt , xerr ) - This do_step method is needed to fulfill the Error Stepper concept. The state is updated in-place and an error estimate is calculated. A type modelling a Boost.Range can be used for x.
- do_step( sys , x , dxdt , t , dt , xerr ) - This method updates the state in-place, but the derivative at the point t must be passed in dxdt. An error estimate is calculated.
- do_step( sys , in , t , out , dt , xerr ) - This method updates the state out-of-place and estimates the error during the step.
• `do_step(sys, in, dxdt, t, out, dt, xerr)` - This method updates the state out-of-place and estimates the error during the step. Furthermore, the derivative at \( t \) must be passed in \( dxdt \).

**Note**

The system is always passed as value, which might result in poor performance if it contains data. In this case it can be used with `boost::ref` or `std::ref`, for example `stepper.do_step(boost::ref(sys), x, t, dt);`

The time \( t \) is not advanced by the stepper. This has to done manually, or by the appropriate `integrate` routines or iterators.

**Template Parameters**

1. `typename Stepper`

   The stepper on which this class should work. It is used via CRTP, hence `explicit_stepper_base` provides the interface for the Stepper.

2. `unsigned short Order`

   The order of a stepper if the stepper is used without error estimation.

3. `unsigned short StepperOrder`

   The order of a step if the stepper is used with error estimation. Usually Order and StepperOrder have the same value.

4. `unsigned short ErrorOrder`

   The order of the error step if the stepper is used with error estimation.

5. `typename State`

   The state type for the stepper.

6. `typename Value`

   The value type for the stepper. This should be a floating point type, like float, double, or a multiprecision type. It must not necessary be the value_type of the State. For example the State can be a `vector< complex< double > >` in this case the Value must be double. The default value is double.

7. `typename Deriv`

   The type representing time derivatives of the state type. It is usually the same type as the state type, only if used with Boost.Units both types differ.

8. `typename Time`

   The type representing the time. Usually the same type as the value type. When Boost.Units is used, this type has usually a unit.

9. `typename Algebra`
The algebra type which must fulfill the Algebra Concept.

10. `typename Operations`

The type for the operations which must fulfill the Operations Concept.

11. `typename Resizer`

The resizer policy class.

**explicit_error_stepper_base public construct/copy/destruct**

1. `explicit_error_stepper_base(const algebra_type & algebra = algebra_type());`

Constructs a `explicit_error_stepper_base` class. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters: `algebra` A copy of algebra is made and stored inside `explicit_stepper_base`.

**explicit_error_stepper_base public member functions**

1. `order_type order(void) const;`

Returns: Returns the order of the stepper if it used without error estimation.

2. `order_type stepper_order(void) const;`

Returns: Returns the order of a step if the stepper is used without error estimation.

3. `order_type error_order(void) const;`

Returns: Returns the order of an error step if the stepper is used without error estimation.

4. `template<typename System, typename StateInOut> void do_step(System system, StateInOut & x, time_type t, time_type dt);`

This method performs one step. It transforms the result in-place.

Parameters: `dt` The step size. `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept. `t` The value of the time, at which the step should be performed. `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in `x`.

5. `template<typename System, typename StateInOut> void do_step(System system, const StateInOut & x, time_type t, time_type dt);`

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

6. `template<typename System, typename StateInOut, typename DerivIn> boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt);`
The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of \( x \) is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in \( x \). This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other `do_step` versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- \( \text{dt} \) The step size.
- \( \text{dxdt} \) The derivative of \( x \) at \( t \).
- \( \text{system} \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( \text{t} \) The value of the time, at which the step should be performed.
- \( \text{x} \) The state of the ODE which should be solved. After calling `do_step` the result is updated in \( x \).

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if \( \text{StateIn} \) and Time are the same type. In this case the method can not be distinguished from other `do_step` variants.

**Note**

This method does not solve the forwarding problem.

Parameters:
- \( \text{dt} \) The step size.
- \( \text{in} \) The state of the ODE which should be solved. \( \text{in} \) is not modified in this method
- \( \text{out} \) The result of the step is written in \( \text{out} \).
- \( \text{system} \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( \text{t} \) The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of \( x \) at \( t \) is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if \( \text{DerivIn} \) and Time are of same type.
Note
This method does not solve the forwarding problem.

Parameters:

- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

```
9. template<typename System, typename StateInOut, typename Err>
    void do_step(System system, StateInOut & x, time_type t, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

Parameters:

- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. x is updated by this method.
- **xerr**: The estimation of the error is stored in xerr.

```
10. template<typename System, typename StateInOut, typename Err>
    void do_step(System system, const StateInOut & x, time_type t, time_type dt, Err & xerr);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

```
11. template<typename System, typename StateInOut, typename DerivIn, typename Err>
    boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
    do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt , xerr );
```

The result is updated in place in x. This method is disabled if Time and DerivIn are of the same type. In this case the method could not be distinguished from other do_step versions.

Note
This method does not solve the forwarding problem.

Parameters:

- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.
- **xerr**: The error estimate is stored in xerr.
12. template<typename System, typename StateIn, typename StateOut, typename Err>
    void do_step(System system, const StateIn & in, time_type t, StateOut & out,
                  time_type dt, Err & xerr);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

**Note**
This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

13. template<typename System, typename StateIn, typename DerivIn,
          typename StateOut, typename Err>
    void do_step(System system, const StateIn & in, const DerivIn & dxdt,
                 time_type t, StateOut & out, time_type dt, Err & xerr);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated. It is supposed to be used in the following way:

```cpp
sys( in, dxdt, t );
stepper.do_step( sys, in, dxdt, t, out, dt );
```

This method is disabled if DerivIn and Time are of same type.

**Note**
This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

14. template<typename StateIn> void adjust_size(const StateIn & x);

Adjust the size of all temporaries in the stepper manually.

Parameters: **x** A state from which the size of the temporaries to be resized is deduced.

15. algebra_type & algebra();

Returns: A reference to the algebra which is held by this class.
const algebra_type & algebra() const;

Returns: A const reference to the algebra which is held by this class.

**explicit_error_stepper_base private member functions**

1. `template<typename System, typename StateInOut>
   void do_step_v1(System system, StateInOut & x, time_type t, time_type dt);

2. `template<typename System, typename StateInOut, typename Err>
   void do_step_v5(System system, StateInOut & x, time_type t, time_type dt,
                   Err & xerr);

3. `template<typename StateIn> bool resize_impl(const StateIn & x);

4. `stepper_type & stepper(void);

5. `const stepper_type & stepper(void) const;

**Header** `<boost/numeric/odeint/stepper/base/explicit_error_stepper_fsal_base.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper, unsigned short Order,
                     unsigned short StepperOrder, unsigned short ErrorOrder,
                     typename State, typename Value, typename Deriv, typename Time,
                     typename Algebra, typename Operations, typename Resizer>
            class explicit_error_stepper_fsal_base;
        }
    }
}
```

**Class template explicit_error_stepper_fsal_base**

boost::numeric::odeint::explicit_error_stepper_fsal_base — Base class for explicit steppers with error estimation and stepper fulfilling the FSAL (first-same-as-last) property. This class can be used with controlled steppers for step size control.
Synopsis

```cpp
#include <boost/numeric/odeint/stepper/base/explicit_error_stepper_fsal_base.hpp>

namespace boost { namespace numeric { namespace odeint { namespace algebra_stepper_base {

/// In header: <boost/numeric/odeint/stepper/base/explicit_error_stepper_fsal_base.hpp>

template<typename Stepper, unsigned short Order, unsigned short StepperOrder,
         unsigned short ErrorOrder, typename State, typename Value,
         typename Deriv, typename Time, typename Algebra, typename Operations,
         typename Resizer>
class explicit_error_stepper_fsal_base :
public boost::numeric::odeint::algebra_stepper_base< Algebra, Operations >
{
public:
  // types
  typedef algebra_stepper_base< Algebra, Operations > algebra_stepper_base_type;
  typedef algebra_stepper_base_type::algebra_type algebra_type;
  typedef State state_type;
  typedef Value value_type;
  typedef Deriv deriv_type;
  typedef Time time_type;
  typedef Resizer resizer_type;
  typedef Stepper stepper_type;
  typedef explicit_error_stepper_fsal_tag stepper_category;
  typedef unsigned short order_type;

  // construct/copy/destruct
  explicit_error_stepper_fsal_base(const algebra_type & = algebra_type());

  // public member functions
  order_type order(void) const;
  order_type stepper_order(void) const;
  order_type error_order(void) const;
  template<typename System, typename StateInOut>
  void do_step(System, StateInOut &, time_type, time_type);
  template<typename System, typename StateInOut>
  void do_step(System, const StateInOut &, time_type, time_type);
  template<typename System, typename StateInOut, typename DerivInOut>
  boost::disable_if< boost::is_same< StateInOut, time_type >, void >::type
do_step(System, StateInOut &, DerivInOut &, time_type, time_type);
  template<typename System, typename StateIn, typename StateOut>
  void do_step(System, const StateIn &, time_type, StateOut &, time_type);
  template<typename System, typename StateIn, typename DerivIn,
           typename StateOut, typename DerivOut, typename Err>
do_step(System, const StateIn &, const DerivIn &, time_type,
           StateOut &, DerivOut &, time_type, Err &);
  template<typename StateIn> void adjust_size(const StateIn &);
  void reset(void);

};
}}}}
```

Boost.Numeric.Odeint

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template<typename DerivIn> void initialize(const DerivIn &);
template<typename System, typename StateIn>
void initialize(System, const StateIn &, time_type);
bool is_initialized(void) const;
algebra_type & algebra();
const algebra_type & algebra() const;

// private member functions
template<typename System, typename StateInOut>
void do_step_v1(System, StateInOut &, time_type, time_type);
template<typename System, typename StateInOut, typename Err>
void do_step_v5(System, StateInOut &, time_type, time_type, Err &);
stepper_type & stepper(void);
const stepper_type & stepper(void) const;

// public data members
static const order_type order_value;
static const order_type stepper_order_value;
static const order_type error_order_value;
};

### Description

This class serves as the base class for all explicit steppers with algebra and operations and which fulfill the FSAL property. In contrast to `explicit_stepper_base` it also estimates the error and can be used in a controlled stepper to provide step size control.

The FSAL property means that the derivative of the system at \( t+dt \) is already used in the current step going from \( t \) to \( t+dt \). Therefore, some more do_steps method can be introduced and the controlled steppers can explicitly make use of this property.

#### Note

This stepper provides do_step methods with and without error estimation. It has therefore three orders, one for the order of a step if the error is not estimated. The other two orders are the orders of the step and the error step if the error estimation is performed.

### Note

This stepper provides do_step methods with and without error estimation. It has therefore three orders, one for the order of a step if the error is not estimated. The other two orders are the orders of the step and the error step if the error estimation is performed.

`explicit_error_stepper_fsal_base` is used as the interface in a CRTP (currently recurring template pattern). In order to work correctly the parent class needs to have a method `do_step_impl( system , in , dxdt_in , t , out , dxdt_out , dt , xerr )`. `explicit_error_stepper_fsal_base` derives from `algebra_stepper_base`.

This class can have an intrinsic state depending on the explicit usage of the do_step method. This means that some do_step methods are expected to be called in order. For example the `do_step( sys , x , t , dt , xerr )` will keep track of the derivative of \( x \) which is the internal state. The first call of this method is recognized such that one does not explicitly initialize the internal state, so it is safe to use this method like

```cpp
stepper_type stepper;
stepper.do_step( sys , x , t , dt , xerr );
stepper.do_step( sys , x , t , dt , xerr );
stepper.do_step( sys , x , t , dt , xerr );
```

But it is unsafe to call this method with different system functions after each other. Do do so, one must initialize the internal state with the `initialize` method or reset the internal state with the `reset` method.

`explicit_error_stepper_fsal_base` provides several overloaded do_step methods, see the list below. Only two of them are needed to fulfill the Error Stepper concept. The other ones are for convenience and for better performance. Some of them simply update the state out-of-place, while other expect that the first derivative at \( t \) is passed to the stepper.
• `do_step(sys, x, t, dt)` - The classical `do_step` method needed to fulfill the Error Stepper concept. The state is updated in-place. A type modelling a Boost.Range can be used for `x`.

• `do_step(sys, x, dxdt, t, dt)` - This method updates the state `x` and the derivative `dxdt` in-place. It is expected that `dxdt` has the value of the derivative of `x` at time `t`.

• `do_step(sys, in, t, out, dt)` - This method updates the state out-of-place, hence the result of the step is stored in `out`.

• `do_step(sys, in, dxdt_in, t, out, dxdt_out, dt)` - This method updates the state and the derivative out-of-place. It expects that the derivative at the point `t` is explicitly passed in `dxdt_in`.

• `do_step(sys, x, t, dt, xerr)` - This `do_step` method is needed to fulfill the Error Stepper concept. The state is updated in-place and an error estimate is calculated. A type modelling a Boost.Range can be used for `x`.

• `do_step(sys, x, dxdt, t, dt, xerr)` - This method updates the state and the derivative in-place. It is assumed that the `dxdt` has the value of the derivative of `x` at time `t`. An error estimate is calculated.

• `do_step(sys, in, t, out, dt, xerr)` - This method updates the state out-of-place and estimates the error during the step.

• `do_step(sys, in, dxdt_in, t, out, dxdt_out, dt, xerr)` - This methods updates the state and the derivative out-of-place and estimates the error during the step. It is assumed the `dxdt_in` is derivative of `in` at time `t`.

**Note**

The system is always passed as value, which might result in poor performance if it contains data. In this case it can be used with `boost::ref` or `std::ref`, for example `stepper.do_step( boost::ref(sys), x, t, dt );`

The time `t` is not advanced by the stepper. This has to done manually, or by the appropriate `integrate routines` or `iterators`.

**Template Parameters**

1. `typename Stepper`

   The stepper on which this class should work. It is used via CRTP, hence `explicit_stepper_base` provides the interface for the Stepper.

2. `unsigned short Order`

   The order of a stepper if the stepper is used without error estimation.

3. `unsigned short StepperOrder`

   The order of a step if the stepper is used with error estimation. Usually `Order` and `StepperOrder` have the same value.

4. `unsigned short ErrorOrder`

   The order of the error step if the stepper is used with error estimation.

5. `typename State`

   The state type for the stepper.
6. **typename Value**

   The value type for the stepper. This should be a floating point type, like float, double, or a multiprecision type. It must not necessarily be the `value_type` of the State. For example, the State can be a `vector< complex< double > >` in this case the Value must be double. The default value is double.

7. **typename Deriv**

   The type representing time derivatives of the state type. It is usually the same type as the state type, only if used with Boost.Units both types differ.

8. **typename Time**

   The type representing the time. Usually the same type as the value type. When Boost.Units is used, this type has usually a unit.

9. **typename Algebra**

   The algebra type which must fulfill the Algebra Concept.

10. **typename Operations**

    The type for the operations which must fulfill the Operations Concept.

11. **typename Resizer**

    The resizer policy class.

**explicit_error_stepper_fsal_base** public construct/copy/destroy

1. **explicit_error_stepper_fsal_base(const algebra_type & algebra = algebra_type());**

   Constructs a `explicit_stepper_fsal_base` class. This constructor can be used as a default constructor if the algebra has a default constructor.

   Parameters:  
   - `algebra`  
     A copy of algebra is made and stored inside `explicit_stepper_base`.

**explicit_error_stepper_fsal_base** public member functions

1. **order_type order(void) const;**

   Returns:  
   Returns the order of the stepper if it used without error estimation.

2. **order_type stepper_order(void) const;**

   Returns:  
   Returns the order of a step if the stepper is used without error estimation.

3. **order_type error_order(void) const;**

   Returns:  
   Returns the order of an error step if the stepper is used without error estimation.
4. 

\[
\begin{array}{ll}
t\text{emplate<typename } & \text{System, typename } \text{StateInOut>}
\end{array}
\]

\[
\begin{array}{ll}
\text{void do_step(System system, StateInOut & x, time_type t, time_type dt);}
\end{array}
\]

This method performs one step. It transforms the result in-place.

**Note**

This method uses the internal state of the stepper.

Parameters:

- **dt** The step size.
- **system** The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **x** The state of the ODE which should be solved. After calling do_step the result is updated in x.

5. 

\[
\begin{array}{ll}
t\text{emplate<typename } & \text{System, typename } \text{StateInOut>}
\end{array}
\]

\[
\begin{array}{ll}
\text{void do_step(System system, const StateInOut & x, time_type t, time_type dt);}
\end{array}
\]

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

6. 

\[
\begin{array}{ll}
t\text{emplate<typename } & \text{System, typename } \text{StateInOut, typename } \text{DerivInOut>}
\end{array}
\]

\[
\begin{array}{ll}
\text{boost::disable_if< boost::is_same< StateInOut, time_type >, void >::type do_step(System system, StateInOut & x, DerivInOut & dxdt, time_type t, time_type dt);}
\end{array}
\]

The method performs one step with the stepper passed by Stepper. Additionally to the other methods the derivative of x is also passed to this method. Therefore, \( dxdt \) must be evaluated initially:

\[
\begin{array}{ll}
\text{ode( x , dxdt , t );}
\end{array}
\]

\[
\begin{array}{ll}
\text{for( ... )}
\end{array}
\]

\[
\begin{array}{ll}
\{ 
\text{stepper.do_step( ode , x , dxdt , t , dt );}
\end{array}
\]

\[
\begin{array}{ll}
t += dt;
\end{array}
\]

**Note**

This method does NOT use the initial state, since the first derivative is explicitly passed to this method.

The result is updated in place in x as well as the derivative \( dxdt \). This method is disabled if Time and StateInOut are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:

- **dt** The step size.
- **dxdt** The derivative of x at t. After calling do_step \( dxdt \) is updated to the new value.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **x** The state of the ODE which should be solved. After calling do_step the result is updated in x.
The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other do_step variants.

**Note**

This method uses the internal state of the stepper.

This method does not solve the forwarding problem.

Parameters:

- **dt**  The step size.
- **in**  The state of the ODE which should be solved. in is not modified in this method
- **out**  The result of the step is written in out.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and updated by the stepper to its new value at t+dt.

**Note**

This method does not solve the forwarding problem.

This method does NOT use the internal state of the stepper.

Parameters:

- **dt**  The step size.
- **dxdt_in**  The derivative of x at t.
- **dxdt_out**  The updated derivative of out at t+dt.
- **in**  The state of the ODE which should be solved. in is not modified in this method
- **out**  The result of the step is written in out.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

**Note**

This method uses the internal state of the stepper.
Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>dt</code></td>
<td>The step size.</td>
</tr>
<tr>
<td><code>system</code></td>
<td>The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.</td>
</tr>
<tr>
<td><code>t</code></td>
<td>The value of the time, at which the step should be performed.</td>
</tr>
<tr>
<td><code>x</code></td>
<td>The state of the ODE which should be solved. x is updated by this method.</td>
</tr>
<tr>
<td><code>xerr</code></td>
<td>The estimation of the error is stored in xerr.</td>
</tr>
</tbody>
</table>

10. `template<typename System, typename StateInOut, typename Err>
    void do_step(System system, const StateInOut & x, time_type t, time_type dt,
                   Err & xerr);

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

11. `template<typename System, typename StateInOut, typename DerivInOut, 
        typename Err>
    boost::disable_if< boost::is_same< StateInOut, time_type >, void >::type
    do_step(System system, StateInOut & x, DerivInOut & dxdt, time_type t,
               time_type dt, Err & xerr);

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method and updated by this method.

Note

This method does NOT use the internal state of the stepper.

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other `do_step` versions. This method is disabled if StateInOut and Time are of the same type.

Note

This method does NOT use the internal state of the stepper.

This method does not solve the forwarding problem.

Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>dt</code></td>
<td>The step size.</td>
</tr>
<tr>
<td><code>dxdt</code></td>
<td>The derivative of x at t. After calling <code>do_step</code> this value is updated to the new value at t+dt.</td>
</tr>
<tr>
<td><code>system</code></td>
<td>The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.</td>
</tr>
<tr>
<td><code>t</code></td>
<td>The value of the time, at which the step should be performed.</td>
</tr>
<tr>
<td><code>x</code></td>
<td>The state of the ODE which should be solved. After calling <code>do_step</code> the result is updated in x.</td>
</tr>
<tr>
<td><code>xerr</code></td>
<td>The error estimate is stored in xerr.</td>
</tr>
</tbody>
</table>

12. `template<typename System, typename StateIn, typename StateOut, typename Err>
    void do_step(System system, const StateIn & in, time_type t, StateOut & out,
                  time_type dt, Err & xerr);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

Note

This method uses the internal state of the stepper.

This method does not solve the forwarding problem.
Parameters:

- \( \text{dt} \): The step size.
- \( \text{in} \): The state of the ODE which should be solved. \( \text{in} \) is not modified in this method.
- \( \text{out} \): The result of the step is written in \( \text{out} \).
- \( \text{system} \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( \text{t} \): The value of the time, at which the step should be performed.
- \( \text{xerr} \): The error estimate.

```
template<typename System, typename StateIn, typename DerivIn, 
          typename StateOut, typename DerivOut, typename Err>
void do_step(System system, const StateIn & in, const DerivIn & dxdt_in, 
             time_type t, StateOut & out, DerivOut & dxdt_out, 
             time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of \( x \) at \( t \) is passed to the stepper and the error is estimated.

**Note**

This method does NOT use the internal state of the stepper.

This method does not solve the forwarding problem.

Parameters:

- \( \text{dxdt_in} \): The derivative of \( x \) at \( t \).
- \( \text{dxdt_out} \): The new derivative at \( t+\text{dt} \) is written into this variable.
- \( \text{in} \): The state of the ODE which should be solved. \( \text{in} \) is not modified in this method.
- \( \text{out} \): The result of the step is written in \( \text{out} \).
- \( \text{system} \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( \text{t} \): The value of the time, at which the step should be performed.
- \( \text{xerr} \): The error estimate.

```
template<typename StateIn> void adjust_size(const StateIn & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters:

- \( x \): A state from which the size of the temporaries to be resized is deduced.

```
void reset(void);
```

Resets the internal state of this stepper. After calling this method it is safe to use all do_step method without explicitly initializing the stepper.

```
template<typename DerivIn> void initialize(const DerivIn & deriv);
```

Initializes the internal state of the stepper.

Parameters:

- \( \text{deriv} \): The derivative of \( x \). The next call of do_step expects that the derivative of \( x \) passed to do_step has the value of \( \text{deriv} \).

```
template<typename System, typename StateIn>
void initialize(System system, const StateIn & x, time_type t);
```

Initializes the internal state of the stepper.
This method is equivalent to

\begin{verbatim}
Deriv dxdt;
    system( x, dxdt, t );
    stepper.initialize( dxdt );
\end{verbatim}

Parameters:
- \texttt{system}  The system function for the next calls of \texttt{do_step}.
- \texttt{t}       The current time of the ODE.
- \texttt{x}      The current state of the ODE.

\begin{verbatim}
bool is_initialized(void) const;
\end{verbatim}

Returns if the stepper is already initialized. If the stepper is not initialized, the first call of \texttt{do_step} will initialize the state of the stepper. If the stepper is already initialized the system function can not be safely exchanged between consecutive \texttt{do_step} calls.

\begin{verbatim}
algebra_type & algebra();
\end{verbatim}

Returns: A reference to the algebra which is held by this class.

\begin{verbatim}
const algebra_type & algebra() const;
\end{verbatim}

Returns: A const reference to the algebra which is held by this class.

**explicit_error_stepper_fsal_base private member functions**

1. \begin{verbatim}
\template<typename System, typename StateInOut>
    void do_step_v1(System system, StateInOut & x, time_type t, time_type dt);
\end{verbatim}

2. \begin{verbatim}
\template<typename System, typename StateInOut, typename Err>
    void do_step_v5(System system, StateInOut & x, time_type t, time_type dt,
                     Err & xerr);
\end{verbatim}

3. \begin{verbatim}
\template<typename StateIn>
    bool resize_impl(const StateIn & x);
\end{verbatim}

4. \begin{verbatim}
    stepper_type & stepper(void);
\end{verbatim}

5. \begin{verbatim}
    const stepper_type & stepper(void) const;
\end{verbatim}
Class template explicit_stepper_base

boost::numeric::odeint::explicit_stepper_base — Base class for explicit steppers without step size control and without dense output.
**Synopsis**

```cpp
// In header: <boost/numeric/odeint/stepper/base/explicit_stepper_base.hpp>
template<typename Stepper, unsigned short Order, typename State,
        typename Value, typename Deriv, typename Time, typename Algebra,
        typename Operations, typename Resizer>
class explicit_stepper_base :
    public boost::numeric::odeint::algebra_stepper_base< Algebra, Operations >
{
    public:
        // types
        typedef State state_type;
        typedef Value value_type;
        typedef Deriv deriv_type;
        typedef Time time_type;
        typedef Resizer resizer_type;
        typedef Stepper stepper_type;
        typedef stepper_tag stepper_category;
        typedef algebra_stepper_base< Algebra, Operations > algebra_stepper_base_type;
        typedef algebra_stepper_base_type::algebra_type algebra_type;
        typedef algebra_stepper_base_type::operations_type operations_type;
        typedef unsigned short order_type;

        // construct/copy/destroy
        explicit_stepper_base(const algebra_type & = algebra_type());

        // public member functions
        order_type order(void) const;
        template<typename System, typename StateInOut>
        void do_step(System, StateInOut &, time_type, time_type);
        template<typename System, typename StateInOut>
        void do_step(System, const StateInOut &, time_type, time_type);
        template<typename System, typename StateInOut, typename DerivIn>
        boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
        do_step(System, StateInOut &, const DerivIn &, time_type, time_type);
        template<typename System, typename StateIn, typename StateOut>
        void do_step(System, const StateIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut>
        void do_step(System, const StateIn &, const DerivIn &, time_type,
                     StateOut &, time_type);
        template<typename StateIn> void adjust_size(const StateIn &);
        algebra_type & algebra();
        const algebra_type & algebra() const;

        // private member functions
        stepper_type & stepper(void);
        const stepper_type & stepper(void) const;
        template<typename StateIn> bool resize_impl(const StateIn &);
        template<typename System, typename StateInOut>
        void do_step_v1(System, StateInOut &, time_type, time_type);

        // public data members
        static const order_type order_value;
    }
```

**Description**

This class serves as the base class for all explicit steppers with algebra and operations. Step size control and error estimation as well as dense output are not provided. `explicit_stepper_base` is used as the interface in a CRTP (currently recurring template pattern). In order to work correctly the parent class needs to have a method `do_step_impl`.
This method is used by `explicit_stepper_base`. `explicit_stepper_base` derives from `algebra_stepper_base`. An example how this class can be used is:

```cpp
template< class State , class Value , class Deriv , class Time , class Algebra , class Operations , class Resizer >
class custom_euler : public explicit_stepper_base< 1 , State , Value , Deriv , Time , Algebra , Operations , Resizer >
{
    public:
        typedef explicit_stepper_base< 1 , State , Value , Deriv , Time , Algebra , Operations , Resizer > base_type;
    
    custom_euler( const Algebra &algebra = Algebra() ) { }

    template< class Sys , class StateIn , class DerivIn , class StateOut >
    void do_step_impl( Sys sys , const StateIn &in , const DerivIn &dxdt , Time t , StateOut &out , Time dt )
    {
        m_algebra.for_each3( out , in , dxdt , Operations::scale_sum2< Value , Time >( 1.0 , dt );
    }

    template< class State >
    void adjust_size( const State &x )
    {
        base_type::adjust_size( x );
    }
};
```

For the Stepper concept only the `do_step(sys, x, t, dt)` needs to be implemented. But this class provides additional `do_step` variants since the stepper is explicit. These methods can be used to increase the performance in some situation, for example if one needs to analyze `dxdt` during each step. In this case one can use:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt ); // the value of dxdt is used here
```

The time `t` is not advanced by the stepper. This has to done manually, or by the appropriate `integrate` routines or iterators.

Note

The system is always passed as value, which might result in poor performance if it contains data. In this case it can be used with `boost::ref` or `std::ref`, for example:

```cpp
stepper.do_step( boost::ref( sys ) , x , t , dt );
```

The time `t` is not advanced by the stepper. This has to done manually, or by the appropriate `integrate` routines or iterators.
**Template Parameters**

1. **typename Stepper**

   The stepper on which this class should work. It is used via CRTP, hence `explicit_stepper_base` provides the interface for the Stepper.

2. **unsigned short Order**

   The order of the stepper.

3. **typename State**

   The state type for the stepper.

4. **typename Value**

   The value type for the stepper. This should be a floating point type, like float, double, or a multiprecision type. It must not necessarily be the `value_type` of the State. For example the State can be a `vector< complex< double > >` in this case the Value must be double. The default value is double.

5. **typename Deriv**

   The type representing time derivatives of the state type. It is usually the same type as the state type, only if used with Boost.Units both types differ.

6. **typename Time**

   The type representing the time. Usually the same type as the value type. When Boost.Units is used, this type has usually a unit.

7. **typename Algebra**

   The algebra type which must fulfill the Algebra Concept.

8. **typename Operations**

   The type for the operations which must fulfill the Operations Concept.

9. **typename Resizer**

   The resizer policy class.

---

**explicit_stepper_base public construct/copy/destroy**

1. `explicit_stepper_base(const algebra_type & algebra = algebra_type());`

   Constructs a `explicit_stepper_base` class. This constructor can be used as a default constructor if the algebra has a default constructor.

   Parameters: `algebra` A copy of algebra is made and stored inside `explicit_stepper_base`. 
explicit_stepper_base public member functions

1. order_type order(void) const;

Returns: Returns the order of the stepper.

2. template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);

This method performs one step. It transforms the result in-place.

Parameters:
- dt The step size.
- system The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- t The value of the time, at which the step should be performed.
- x The state of the ODE which should be solved. After calling do_step the result is updated in x.

3. template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

4. template<typename System, typename StateInOut, typename DerivIn>
   boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
   do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt);

The method performs one step. Additionally to the other method the derivative of \( x \) is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in \( x \). This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

Note

This method does not solve the forwarding problem.

Parameters:
- dt The step size.
- dxdt The derivative of \( x \) at \( t \).
- system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- t The value of the time, at which the step should be performed.
- x The state of the ODE which should be solved. After calling do_step the result is updated in \( x \).

5. template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);

The method performs one step. The state of the ODE is updated out-of-place.
**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt`: The step size.
- `in`: The state of the ODE which should be solved. `in` is not modified in this method.
- `out`: The result of the step is written in `out`.
- `system`: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t`: The value of the time, at which the step should be performed.

```cpp
template<typename System, typename StateIn, typename DerivIn, typename StateOut>
void do_step(System system, const StateIn & in, const DerivIn & dxdt, time_type t, StateOut & out, time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of \( x \) at \( t \) is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt`: The step size.
- `dxdt`: The derivative of \( x \) at \( t \).
- `in`: The state of the ODE which should be solved. `in` is not modified in this method.
- `out`: The result of the step is written in `out`.
- `system`: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t`: The value of the time, at which the step should be performed.

```cpp
template<typename StateIn> void adjust_size(const StateIn & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters: `x` A state from which the size of the temporaries to be resized is deduced.

```cpp
algebra_type & algebra();
```

Returns: A reference to the algebra which is held by this class.

```cpp
const algebra_type & algebra() const;
```

Returns: A const reference to the algebra which is held by this class.

**explicit_stepper_base** private member functions

1. `stepper_type & stepper(void);`
2. `const stepper_type & stepper(void) const;`
3. `template<typename StateIn> bool resize_impl(const StateIn & x);`

4. `template<typename System, typename StateInOut>
   void do_step_v1(System system, StateInOut & x, time_type t, time_type dt);`

Header `<boost/numeric/odeint/stepper/base/symplectic_rkn_stepper_base.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<size_t NumOfStages, unsigned short Order, typename Coor,
                     typename Momentum, typename Value, typename CoorDeriv,
                     typename MomentumDeriv, typename Time, typename Algebra,
                     typename Operations, typename Resizer>
            class symplectic_nystroem_stepper_base;
        }
    }
}
```

Class template `symplectic_nystroem_stepper_base`

boost::numeric::odeint::symplectic_nystroem_stepper_base — Base class for all symplectic steppers of Nystroem type.
Synopsis

// In header: <boost/numeric/odeint/stepper/base/symplectic_rkn_stepper_base.hpp>

template<size_t NumOfStages, unsigned short Order, typename Coor,
         typename Momentum, typename Value, typename CoorDeriv,
         typename MomentumDeriv, typename Time, typename Algebra,
         typename Operations, typename Resizer>
class symplectic_nystroem_stepper_base :
public boost::numeric::odeint::algebra_stepper_base< Algebra, Operations >

public:
// types
typedef algebra_stepper_base< Algebra, Operations > algebra_stepper_base_type;
typedef algebra_stepper_base_type::algebra_type algebra_type;
typedef algebra_stepper_base_type::operations_type operations_type;
typedef Coor coor_type;
typedef Momentum momentum_type;
typedef std::pair< coor_type, momentum_type > state_type;
typedef CoorDeriv coor_deriv_type;
typedef state_wrapper< coor_deriv_type > wrapped_coor_deriv_type;
typedef MomentumDeriv momentum_deriv_type;
typedef state_wrapper< momentum_deriv_type > wrapped_momentum_deriv_type;
typedef std::pair< coor_deriv_type, momentum_deriv_type > deriv_type;
typedef Value value_type;
typedef Time time_type;
typedef Resizer resizer_type;
typedef stepper_tag stepper_category;
typedef unsigned short order_type;
typedef boost::array< value_type, num_of_stages > coef_type;

// construct/copy/destruct
symplectic_nystroem_stepper_base(const coef_type &, const coef_type &,
const algebra_type & = algebra_type());

// public member functions
order_type order(void) const;
template<typename System, typename StateInOut>
void do_step(System, const StateInOut &, time_type, time_type);
template<typename System, typename StateIn, typename StateOut>
void do_step_impl(System, const StateIn &, time_type, StateOut &,
boost::mpl::true_);
template<typename System, typename StateIn, typename StateOut>
void do_step_impl(System, const StateIn &, time_type, StateOut &,
boost::mpl::false_);
template<typename StateIn> bool resize_dqdt(const StateIn &);

// private member functions
template<typename System, typename StateIn, typename StateOut>
void do_step_impl(System, const StateIn &, time_type, StateOut &,
boost::mpl::true_);
template<typename System, typename StateIn, typename StateOut>
void do_step_impl(System, const StateIn &, time_type, StateOut &,
boost::mpl::false_);
template<typename StateIn> bool resize_dqdt(const StateIn &);

template<typename StateIn> bool resize_dpdt(const StateIn &);

// public data members
static const size_t num_of_stages;
static const order_type order_value;
}

Description

This class is the base class for the symplectic Runge-Kutta-Nystroem steppers. Symplectic steppers are usually used to solve Hamiltonian systems and they conserve the phase space volume, see en.wikipedia.org/wiki/Symplectic_integrator. Furthermore, the energy is conserved in average. In detail this class of steppers can be used to solve separable Hamiltonian systems which can be written in the form $H(q,p) = H_1(p) + H_2(q)$. $q$ is usually called the coordinate, while $p$ is the momentum. The equations of motion are $dq/dt = dH_1/dp$, $dp/dt = -dH_2/dq$.

ToDo: add formula for solver and explanation of the coefficients.

symplectic_nystroem_stepper_base uses odeint's algebra and operation system. Step size and error estimation are not provided for this class of solvers. It derives from algebra_stepper_base. Several do_step variants are provided:

- **do_step(sys, x, t, dt)** - The classical do_step method. The sys can be either a pair of function objects for the coordinate or the momentum part or one function object for the momentum part. $x$ is a pair of coordinate and momentum. The state is updated in-place.

- **do_step(sys, q, p, t, dt)** - This method is similar to the method above with the difference that the coordinate and the momentum are passed explicitly and not packed into a pair.

- **do_step(sys, x_in, t, x_out, dt)** - This method transforms the state out-of-place. $x_{in}$ and $x_{out}$ are here pairs of coordinate and momentum.

**Template Parameters**

1. size_t NumOfStages
   Number of stages.

2. unsigned short Order
   The order of the stepper.

3. typename Coor
   The type representing the coordinates $q$.

4. typename Momentum
   The type representing the coordinates $p$.

5. typename Value
   The basic value type. Should be something like float, double or a high-precision type.

6. typename CoorDeriv
   The type representing the time derivative of the coordinate $dq/dt$. 

---

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The type representing the time \( t \).

The algebra.

The operations.

The resizer policy.

```cpp
symplectic_nystroem_stepper_base public construct/copy/destroy

symplectic_nystroem_stepper_base(const coef_type & coef_a,
    const coef_type & coef_b,
    const algebra_type & algebra = algebra_type());
```

Constructs a `symplectic_nystroem_stepper_base` class. The parameters of the specific Nystroem method and the algebra have to be passed.

- **Parameters:**
  - `algebra`: A copy of algebra is made and stored inside `explicit_stepper_base`.
  - `coef_a`: The coefficients \( a \).
  - `coef_b`: The coefficients \( b \).

`symplectic_nystroem_stepper_base public member functions`

```cpp
order_type order(void) const;
```

Returns: Returns the order of the stepper.

```cpp
template<typename System, typename StateInOut>
void do_step(System system, const StateInOut & state, time_type t,
    time_type dt);
```

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( \frac{dq}{dt} = p \). The state is updated in-place.

**Note**

boost::ref or std::ref can be used for the system as well as for the state. So, it is correct to write:

```cpp
step_per.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , make_pair( std::ref( q ) , std::ref( p ) ) , t , dt ).
```

This method solves the forwarding problem.
Parameters: 

- **dt**
  - The time step.
- **state**
  - The state of the ODE. It is a pair of Coor and Momentum. The state is updated in-place, therefore, the new value of the state will be written into this variable.
- **system**
  - The system, can be represented as a pair of two function object or one function object. See above.
- **t**
  - The time of the ODE. It is not advanced by this method.

3. 

    template<typename System, typename StateInOut>
    void do_step(System system, StateInOut & state, time_type t, time_type dt);

Same function as above. It differs only in a different const specifier in order to solve the forwarding problem, can be used with Boost.Range.

4. 

    template<typename System, typename CoorInOut, typename MomentumInOut>
    void do_step(System system, CoorInOut & q, MomentumInOut & p, time_type t, time_type dt);

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial $dq/dt = p$. The state is updated in-place.

---

**Note**

boost::ref or std::ref can be used for the system. So, it is correct to write stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , q , p , t , dt ).

This method solves the forwarding problem.

Parameters: 

- **dt**
  - The time step.
- **p**
  - The momentum of the ODE. It is updated in-place. Therefore, the new value of the momentum will be written into this variable.
- **q**
  - The coordinate of the ODE. It is updated in-place. Therefore, the new value of the coordinate will be written into this variable.
- **system**
  - The system, can be represented as a pair of two function object or one function object. See above.
- **t**
  - The time of the ODE. It is not advanced by this method.

5. 

    template<typename System, typename CoorInOut, typename MomentumInOut>
    void do_step(System system, const CoorInOut & q, const MomentumInOut & p, time_type t, time_type dt);

Same function as do_step( system , q , p , t , dt ). It differs only in a different const specifier in order to solve the forwarding problem, can be called with Boost.Range.

6. 

    template<typename System, typename StateIn, typename StateOut>
    void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial $dq/dt = p$. The state is updated out-of-place.
Note

boost::ref or std::ref can be used for the system. So, it is correct to write

\[ \text{stepper.do_step( make_pair( std::ref( fq ), std::ref( fp ) ), x_in , t , x_out , dt ).} \]

This method NOT solve the forwarding problem.

Parameters:
- \( \text{dt} \): The time step.
- \( \text{in} \): The state of the ODE, which is a pair of coordinate and momentum. The state is updated out-of-place, therefore the new value is written into \( \text{out} \).
- \( \text{system} \): The system, can be represented as a pair of two function object or one function object. See above.
- \( \text{t} \): The time of the ODE. It is not advanced by this method.

7. 
\[
\text{template<typename StateType> void adjust_size(const StateType & x);} 
\]

Adjust the size of all temporaries in the stepper manually.

Parameters:
- \( x \): A state from which the size of the temporaries to be resized is deduced.

8. 
\[
\text{const coef_type & coef_a(void) const; } 
\]

Returns the coefficients \( a \).

9. 
\[
\text{const coef_type & coef_b(void) const; } 
\]

Returns the coefficients \( b \).

10. 
\[
\text{algebra_type & algebra(); } 
\]

Returns: A reference to the algebra which is held by this class.

11. 
\[
\text{const algebra_type & algebra() const; } 
\]

Returns: A const reference to the algebra which is held by this class.

**symplectic_nystroem_stepper_base private member functions**

1. 
\[
\text{template<typename System, typename StateIn, typename StateOut> void do_step_impl(System system, const StateIn & in, time_type t, StateOut & out, time_type dt, boost::mpl::true_);} 
\]

2. 
\[
\text{template<typename System, typename StateIn, typename StateOut> void do_step_impl(System system, const StateIn & in, time_type, StateOut & out, time_type dt, boost::mpl::false_);} 
\]

3. 
\[
\text{template<typename StateIn> bool resize_dqdt(const StateIn & x);} 
\]

4. 
\[
\text{template<typename StateIn> bool resize_dpdt(const StateIn & x);} 
\]
Class template bulirsch_stoer

boost::numeric::odeint::bulirsch_stoer — The Bulirsch-Stoer algorithm.
Synopsis

// In header: <boost/numeric/odeint/stepper/bulirsch_stoer.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class bulirsch_stoer {
public:
  // types
  typedef State state_type;
  typedef Value value_type;
  typedef Deriv deriv_type;
  typedef Time time_type;
  typedef Algebra algebra_type;
  typedef Operations operations_type;
  typedef Resizer resizer_type;

  // construct/copy/destruct
  bulirsch_stoer(value_type = 1E-6, value_type = 1E-6, value_type = 1.0,
                 value_type = 1.0);

  // public member functions
  template<typename System, typename StateInOut>
  controlled_step_result try_step(System, StateInOut &, time_type &, time_type &);
  template<typename System, typename StateInOut>
  controlled_step_result try_step(System, const StateInOut &, time_type &, time_type &);
  template<typename System, typename StateInOut, typename DerivIn>
  controlled_step_result try_step(System, StateInOut &, const DerivIn &, time_type &, time_type &);
  template<typename System, typename StateIn, typename StateOut>
  boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type
                 try_step(System, const StateIn &, time_type &, StateOut &, time_type &);
  template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
  controlled_step_result try_step(System, const StateIn &, const DerivIn &, time_type &,
                 StateOut &, time_type &);
  void reset();
  template<typename StateIn> void adjust_size(const StateIn &);

  // private member functions
  template<typename StateIn> bool resize_m_dxdt(const StateIn &);
  template<typename StateIn> bool resize_m_xnew(const StateIn &);
  template<typename StateIn> bool resize_impl(const StateIn &);
  template<typename System, typename StateInOut>
  controlled_step_result try_step_v1(System, StateInOut &, time_type &, time_type &);
  template<typename StateInOut>
  void extrapolate(size_t, state_table_type &, const value_matrix &, StateInOut &);
  time_type calc_h_opt(time_type, value_type, size_t) const;
  controlled_step_result set_k_opt(size_t, const inv_time_vector &, const time_vector &, time_type &);
bool in_convergence_window(size_t) const;
bool should_reject(value_type, size_t) const;

// public data members
static const size_t m_k_max;
}

Description

The Bulirsch-Stoer is a controlled stepper that adjusts both step size and order of the method. The algorithm uses the modified midpoint and a polynomial extrapolation compute the solution.

Template Parameters

1. typename State
   The state type.

2. typename Value = double
   The value type.

3. typename Deriv = State
   The type representing the time derivative of the state.

4. typename Time = Value
   The time representing the independent variable - the time.

5. typename Algebra = range_algebra
   The algebra type.

6. typename Operations = default_operations
   The operations type.

7. typename Resizer = initially_resizer
   The resizer policy type.

bulirsch_stoer public construct/copy/destroy

1. bulirsch_stoer(value_type eps_abs = 1E-6, value_type eps_rel = 1E-6,
                 value_type factor_x = 1.0, value_type factor_dxdt = 1.0);

Constructs the bulirsch_stoer class, including initialization of the error bounds.

Parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>eps_abs</td>
<td>Absolute tolerance level.</td>
</tr>
<tr>
<td>eps_rel</td>
<td>Relative tolerance level.</td>
</tr>
<tr>
<td>factor_dxdt</td>
<td>Factor for the weight of the derivative.</td>
</tr>
<tr>
<td>factor_x</td>
<td>Factor for the weight of the state.</td>
</tr>
</tbody>
</table>

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**bulirsch_stoer public member functions**

1. 

```cpp
template<typename System, typename StateInOut>
controlled_step_result
try_step(System system, StateInOut & x, time_type & t, time_type & dt);
```

Tries to perform one step.

This method tries to do one step with step size \( dt \). If the error estimate is to large, the step is rejected and the method returns fail and the step size \( dt \) is reduced. If the error estimate is acceptably small, the step is performed, success is returned and \( dt \) might be increased to make the steps as large as possible. This method also updates \( t \) if a step is performed. Also, the internal order of the stepper is adjusted if required.

**Parameters:**
- \( dt \): The step size. Updated.
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.
- \( x \): The state of the ODE which should be solved. Overwritten if the step is successful.

**Returns:**
- success if the step was accepted, fail otherwise.

2. 

```cpp
template<typename System, typename StateInOut>
controlled_step_result
try_step(System system, const StateInOut & x, time_type & t, time_type & dt);
```

Second version to solve the forwarding problem, can be used with Boost.Range as StateInOut.

3. 

```cpp
template<typename System, typename StateInOut, typename DerivIn>
controlled_step_result
try_step(System system, StateInOut & x, const DerivIn & dxdt, time_type & t,
         time_type & dt);
```

Tries to perform one step.

This method tries to do one step with step size \( dt \). If the error estimate is to large, the step is rejected and the method returns fail and the step size \( dt \) is reduced. If the error estimate is acceptably small, the step is performed, success is returned and \( dt \) might be increased to make the steps as large as possible. This method also updates \( t \) if a step is performed. Also, the internal order of the stepper is adjusted if required.

**Parameters:**
- \( dt \): The step size. Updated.
- \( dxdt \): The derivative of state.
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.
- \( x \): The state of the ODE which should be solved. Overwritten if the step is successful.

**Returns:**
- success if the step was accepted, fail otherwise.

4. 

```cpp
template<typename System, typename StateIn, typename StateOut>
boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type
try_step(System system, const StateIn & in, time_type & t, StateOut & out,
         time_type & dt);
```

Tries to perform one step.

**Note**

This method is disabled if state\_type=\texttt{time\_type} to avoid ambiguity.
This method tries to do one step with step size \( dt \). If the error estimate is too large, the step is rejected and the method returns `fail` and the step size \( dt \) is reduced. If the error estimate is acceptably small, the step is performed, success is returned and \( dt \) might be increased to make the steps as large as possible. This method also updates \( t \) if a step is performed. Also, the internal order of the stepper is adjusted if required.

Parameters:
- \( dt \): The step size. Updated.
- \( in \): The state of the ODE which should be solved.
- \( out \): Used to store the result of the step.
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.

Returns:
- `success` if the step was accepted, `fail` otherwise.

```cpp
template<typename System, typename StateIn, typename DerivIn, typename StateOut>
controlled_step_result
try_step(System system, const StateIn & in, const DerivIn & dxdt, time_type & t, StateOut & out, time_type & dt);
```

5. Tries to perform one step.

6. Resets the internal state of the stepper.

```cpp
void reset();
```

7. Adjust the size of all temporaries in the stepper manually.

```cpp
template<typename StateIn> void adjust_size(const StateIn & x);
```

---

**bulirsch_stoer** private member functions

1. ```cpp
   template<typename StateIn> bool resize_m_dxdt(const StateIn & x);
```

2. ```cpp
   template<typename StateIn> bool resize_m_xnew(const StateIn & x);
```

3. ```cpp
   template<typename StateIn> bool resize_impl(const StateIn & x);
```
4. ```
   template<typename System, typename StateInOut>
   controlled_step_result
   try_step_v1(System system, StateInOut & x, time_type & t, time_type & dt);
```  
5. ```
   template<typename StateInOut>
   void extrapolate(size_t k, state_table_type & table,
                   const value_matrix & coeff, StateInOut & xest);
```  
6. ```
   time_type calc_h_opt(time_type h, value_type error, size_t k) const;
```  
7. ```
   controlled_step_result
   set_k_opt(size_t k, const inv_time_vector & work, const time_vector & h_opt,
             time_type & dt);
```  
8. ```
   bool in_convergence_window(size_t k) const;
```  
9. ```
   bool should_reject(value_type error, size_t k) const;
```  

Header `<boost/numeric/odeint/stepper/bulirsch_stoer_dense_out.hpp>`

```
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                        typename Deriv = State, typename Time = Value,
                        typename Algebra = range_algebra,
                        typename Operations = default_operations,
                        typename Resizer = initially_resizer>
            class bulirsch_stoer_dense_out;
        }
    }
}
```  

Class template bulirsch_stoer_dense_out

boost::numeric::odeint::bulirsch_stoer_dense_out — The Bulirsch-Stoer algorithm.
Synopsis

// In header: <boost/numeric/odeint/stepper/bulirsch_stoer_dense_out.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class bulirsch_stoer_dense_out {
  public:
    // types
    typedef State state_type;
    typedef Value value_type;
    typedef Deriv deriv_type;
    typedef Time time_type;
    typedef Algebra algebra_type;
    typedef Operations operations_type;
    typedef Resizer resizer_type;
    typedef dense_output_stepper_tag stepper_category;

    // construct/copy/destruct
    bulirsch_stoer_dense_out(value_type = 1E-6, value_type = 1E-6,
                             value_type = 1.0, value_type = 1.0, bool = false);

    // public member functions
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut, typename DerivOut>
    controlled_step_result
      try_step(System, const StateIn &, const DerivIn &, time_type &,
               StateOut &, DerivOut &, time_type &);

    template<typename System>
    void initialize(const StateType &, const time_type &, const time_type &);

    template<typename System> std::pair< time_type, time_type > do_step(System);

    template<typename StateOut> void calc_state(time_type, StateOut &) const;

    const state_type & current_state(void) const;

    time_type current_time(void) const;

    const state_type & previous_state(void) const;

    time_type previous_time(void) const;

    time_type current_time_step(void) const;

    void reset();

    template<typename StateIn> void adjust_size(const StateIn &);

    // private member functions
    template<typename StateInOut, typename StateVector>
    void extrapolate(size_t, StateVector &, const value_matrix &, StateInOut &, size_t = 0);

    template<typename StateVector>
    void extrapolate_dense_out(size_t, StateVector &, const value_matrix &, size_t = 0);

    time_type calc_h_opt(time_type, value_type, size_t) const;

    bool in_convergence_window(size_t) const;

    bool should_reject(value_type, size_t) const;

    template<typename StateIn1, typename DerivIn1, typename StateIn2,
             typename DerivIn2>
    value_type prepare_dense_output(int, const StateIn1 &, const DerivIn1 &,
                                     const StateIn2 &, const DerivIn2 &,
                                     time_type);

    template<typename DerivIn>
    void calculateFinite_difference(size_t, size_t, value_type,
                                     const DerivIn &);

    template<typename StateOut>
    void do_interpolation(time_type, StateOut &) const;
Description

The Bulirsch-Stoer is a controlled stepper that adjusts both step size and order of the method. The algorithm uses the modified midpoint and a polynomial extrapolation compute the solution. This class also provides dense output facility.

Template Parameters

1. `typename State`
   
   The state type.

2. `typename Value = double`
   
   The value type.

3. `typename Deriv = State`
   
   The type representing the time derivative of the state.

4. `typename Time = Value`
   
   The time representing the independent variable - the time.

5. `typename Algebra = range_algebra`
   
   The algebra type.

6. `typename Operations = default_operations`
   
   The operations type.

7. `typename Resizer = initially_resizer`
   
   The resizer policy type.
**Bulirsch-Stoer Dense Output**

**Public Construct/Copy/Destruct**

1. `bulirsch_stoer_dense_out(value_type eps_abs = 1E-6, value_type eps_rel = 1E-6, value_type factor_x = 1.0, value_type factor_dxdt = 1.0, bool control_interpolation = false);`

Constructs the *Bulirsch-Stoer* class, including initialization of the error bounds.

**Parameters:**
- `control_interpolation`: Set true to additionally control the error of the interpolation.
- `eps_abs`: Absolute tolerance level.
- `eps_rel`: Relative tolerance level.
- `factor_dxdt`: Factor for the weight of the derivative.
- `factor_x`: Factor for the weight of the state.

**Bulirsch-Stoer Dense Output Public Member Functions**

1. `template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename DerivOut> controlled_step_result try_step(System system, const StateIn & in, const DerivIn & dxdt, time_type & t, StateOut & out, DerivOut & dxdt_new, time_type & dt);`

Tries to perform one step.

This method tries to do one step with step size `dt`. If the error estimate is too large, the step is rejected and the method returns `fail` and the step size `dt` is reduced. If the error estimate is acceptably small, the step is performed, `success` is returned and `dt` might be increased to make the steps as large as possible. This method also updates `t` if a step is performed. Also, the internal order of the stepper is adjusted if required.

**Parameters:**
- `dxdt`: The derivative of state.
- `in`: The state of the ODE which should be solved.
- `out`: Used to store the result of the step.
- `system`: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t`: The value of the time. Updated if the step is successful.

**Returns:**
- `success` if the step was accepted, `fail` otherwise.

2. `template<typename StateType> void initialize(const StateType & x0, const time_type & t0, const time_type & dt0);`

Initializes the dense output stepper.

**Parameters:**
- `dt0`: The initial time step.
- `t0`: The initial time.
- `x0`: The initial state.

3. `template<typename System> std::pair< time_type, time_type > do_step(System system);`

Does one time step. This is the main method that should be used to integrate an ODE with this stepper.
initialize has to be called before using this method to set the initial conditions x,t and the stepsize.

Parameters: system The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
Returns: Pair with start and end time of the integration step.

4. 

\texttt{template<typename StateOut> void calc\_state(time\_type t, StateOut & x) const;}

Calculates the solution at an intermediate point within the last step.
Parameters: 
\( t \) The time at which the solution should be calculated, has to be in the current time interval.
\( x \) The output variable where the result is written into.

5. 

\texttt{const state\_type & current\_state(void) const;}

Returns the current state of the solution.
Returns: The current state of the solution \( x(t) \).

6. 

\texttt{time\_type current\_time(void) const;}

Returns the current time of the solution.
Returns: The current time of the solution \( t \).

7. 

\texttt{const state\_type & previous\_state(void) const;}

Returns the last state of the solution.
Returns: The last state of the solution \( x(t-dt) \).

8. 

\texttt{time\_type previous\_time(void) const;}

Returns the last time of the solution.
Returns: The last time of the solution \( t-dt \).

9. 

\texttt{time\_type current\_time\_step(void) const;}

Returns the current step size.
Returns: The current step size.

10. 

\texttt{void reset();}

Resets the internal state of the stepper.

11. 

\texttt{template<typename StateIn> void adjust\_size(const StateIn & x);} 

Adjust the size of all temporaries in the stepper manually.
Parameters: x A state from which the size of the temporaries to be resized is deduced.

**bulirsch_stoer_dense_out private member functions**

1. `template<typename StateInOut, typename StateVector>
   void extrapolate(size_t k, StateVector & table, const value_matrix & coeff,
   StateInOut & xest, size_t order_start_index = 0);`

2. `template<typename StateVector>
   void extrapolate_dense_out(size_t k, StateVector & table,
   const value_matrix & coeff,
   size_t order_start_index = 0);`

3. `time_type calc_h_opt(time_type h, value_type error, size_t k) const;`

4. `bool in_convergence_window(size_t k) const;`

5. `bool should_reject(value_type error, size_t k) const;`

6. `template<typename StateIn1, typename DerivIn1, typename StateIn2,
   typename DerivIn2>
   value_type prepare_dense_output(int k, const StateIn1 & x_start,
   const DerivIn1 & dxdt_start,
   const StateIn2 &, const DerivIn2 &,
   time_type dt);`

7. `template<typename DerivIn>
   void calculate_finite_difference(size_t j, size_t kappa, value_type fac,
   const DerivIn & dxdt);`

8. `template<typename StateOut>
   void do_interpolation(time_type t, StateOut & out) const;`

9. `template<typename StateIn> bool resize_impl(const StateIn & x);`

10. `state_type & get_current_state(void);`

11. `const state_type & get_current_state(void) const;`

12. `state_type & get_old_state(void);`
const state_type & get_old_state(void) const;

deriv_type & get_current_deriv(void);

const deriv_type & get_current_deriv(void) const;

deriv_type & get_old_deriv(void);

const deriv_type & get_old_deriv(void) const;

void toggle_current_state(void);

Header `<boost/numeric/odeint/stepper/control_runge_kutta.hpp>`

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Value, typename Algebra = range_algebra,
                    typename Operations = default_operations>
                class default_error_checker;

            template<typename ErrorStepper,
                    typename ErrorChecker = default_error_checker< typename ErrorStep
                    per::value_type ,typename ErrorStepper::algebra_type ,typename ErrorStepper::operations_type >,
                    typename Resizer = typename ErrorStepper::resizer_type,
                    typename ErrorStepperCategory = typename ErrorStepper::stepper_category>
                class controlled_runge_kutta;

            template<typename ErrorStepper, typename ErrorChecker, typename Resizer>
                class controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_step
                per_tag>;

            template<typename ErrorStepper, typename ErrorChecker, typename Resizer>
                class controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_step
                per_fsal_tag>;
        }
    }
}

Class template default_error_checker

boost::numeric::odeint::default_error_checker — The default error checker to be used with Runge-Kutta error steppers.
Synopsis

```cpp
/* In header: <boost/numeric/odeint/stepper/controlled_runge_kutta.hpp> */

template<typename Value, typename Algebra = range_algebra,
         typename Operations = default_operations>
class default_error_checker {
public:
    // types
    typedef Value value_type;
    typedef Algebra algebra_type;
    typedef Operations operations_type;

    // construct/copy/destruct
    default_error_checker(value_type eps_abs = static_cast< value_type >(1.0e-6),
                          value_type eps_rel = static_cast< value_type >(1.0e-6),
                          value_type a_x = static_cast< value_type >(1),
                          value_type a_dxdt = static_cast< value_type >(1));

    // public member functions
    template<typename State, typename Deriv, typename Err, typename Time>
    value_type error(const State &, const Deriv &, Err &, Time) const;
    template<typename State, typename Deriv, typename Err, typename Time>
    value_type error(algebra_type &, const State &, const Deriv &, Err &, Time) const;
};
```

Description

This class provides the default mechanism to compare the error estimates reported by Runge-Kutta error steppers with user defined error bounds. It is used by the controlled_runge_kutta steppers.

Template Parameters

1. `typename Value`
   
   The value type.

2. `typename Algebra = range_algebra`
   
   The algebra type.

3. `typename Operations = default_operations`
   
   The operations type.

`default_error_checker` public construct/copy/destroy

1. `default_error_checker(value_type eps_abs = static_cast< value_type >(1.0e-6),
                          value_type eps_rel = static_cast< value_type >(1.0e-6),
                          value_type a_x = static_cast< value_type >(1),
                          value_type a_dxdt = static_cast< value_type >(1));`
default_error_checker public member functions

1. \[
\text{template<typename State, typename Deriv, typename Err, typename Time>}
\text{value_type error(const State & x_old, const Deriv & dxdt_old, Err & x_err, Time dt) const;}
\]

2. \[
\text{template<typename State, typename Deriv, typename Err, typename Time>}
\text{value_type error(algebra_type & algebra, const State & x_old, const Deriv & dxdt_old, Err & x_err, Time dt) const;}
\]

Class template controlled_runge_kutta

boost::numeric::odeint::controlled_runge_kutta

Synopsis

// In header: <boost/numeric/odeint/stepper/controlled_runge_kutta.hpp>

\[
\text{template<typename ErrorStepper, typename ErrorChecker = default_error_checker< typename ErrorStepper::value_type, type< ErrorStepper::algebra_type, typename ErrorStepper::operations_type >, type< ErrorStepper::resizer_type, typename ErrorStepper::stepper_category> >}
\text{class controlled_runge_kutta {}}
\]

Description

Specializations

• Class template controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_stepper_tag>

• Class template controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_stepper_fsal_tag>

Class template controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_stepper_tag>

boost::numeric::odeint::controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_stepper_tag> — Implements step size control for Runge-Kutta steppers with error estimation.
Synopsis

// In header: <boost/numeric/odeint/stepper/controlled_runge_kutta.hpp>

template<typename ErrorStepper, typename ErrorChecker, typename Resizer>
class controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_stepper_tag> { 
public:
    // types
    typedef ErrorStepper             stepper_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::time_type  time_type;
    typedef stepper_type::algebra_type algebra_type;
    typedef stepper_type::operations_type operations_type;
    typedef Resizer                  resizer_type;
    typedef ErrorChecker             error_checker_type;
    typedef explicit_controlled_stepper_tag stepper_category;

    // construct/copy/destruct
    controlled_runge_kutta(const error_checker_type & = error_checker_type(),
                           const stepper_type & = stepper_type());

    // public member functions
    template<typename System, typename StateInOut>
    controlled_step_result try_step(System, StateInOut &, time_type &, time_type &);
    template<typename System, typename StateInOut>
    controlled_step_result try_step(System, const StateInOut &, time_type &, time_type &);
    template<typename System, typename StateInOut, typename DerivIn>
    controlled_step_result try_step(System, StateInOut &, const DerivIn &, time_type &, time_type &);
    template<typename System, typename StateIn, typename StateOut>
    boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type try_step(System, const StateIn &, time_type &, StateOut &, time_type &);
    template<typename System, typename StateIn, typename DerivIn, typename StateOut>
    controlled_step_result try_step(System, const StateIn &, const DerivIn &, time_type &, StateOut &, time_type &);

    value_type last_error(void) const;
    template<typename StateType> void adjust_size(const StateType &);
    stepper_type & stepper(void);
    const stepper_type & stepper(void) const;

    // private member functions
    template<typename System, typename StateInOut>
    controlled_step_result try_step_v1(System, StateInOut &, time_type &, time_type &);
    template<typename System> bool resize_m_xerr_impl(const StateIn &);
    template<typename System> bool resize_m_dxdt_impl(const StateIn &);
    template<typename System> bool resize_m_xnew_impl(const StateIn &); 
};

Description

This class implements the step size control for standard Runge-Kutta steppers with error estimation.
Template Parameters

1. **typename ErrorStepper**
   The stepper type with error estimation, has to fulfill the ErrorStepper concept.

2. **typename ErrorChecker**
   The error checker

3. **typename Resizer**
   The resizer policy type.

**controlled_runge_kutta public construct/copy/destroy**

1. ```cpp
controlled_runge_kutta(const error_checker_type & error_checker = error_checker_type(),
                        const stepper_type & stepper = stepper_type());
```
   Constructs the controlled Runge-Kutta stepper.
   
   Parameters:  
   - `error_checker` An instance of the error checker.  
   - `stepper` An instance of the underlying stepper.

**controlled_runge_kutta public member functions**

1. ```cpp
template<typename System, typename StateInOut>
controlled_step_result
  try_step(System system, StateInOut & x, time_type & t, time_type & dt);
```
   Tries to perform one step.
   
   This method tries to do one step with step size dt. If the error estimate is too large, the step is rejected and the method returns `fail` and the step size dt is reduced. If the error estimate is acceptably small, the step is performed, success is returned and dt might be increased to make the steps as large as possible. This method also updates t if a step is performed.
   
   Parameters:  
   - `dt` The step size. Updated.  
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
   - `t` The value of the time. Updated if the step is successful.  
   - `x` The state of the ODE which should be solved. Overwritten if the step is successful.
   
   Returns:  
   - `success` if the step was accepted, `fail` otherwise.

2. ```cpp
template<typename System, typename StateInOut>
controlled_step_result
  try_step(System system, const StateInOut & x, time_type & t, time_type & dt);
```
   Tries to perform one step. Solves the forwarding problem and allows for using boost range as state_type.
   
   This method tries to do one step with step size dt. If the error estimate is too large, the step is rejected and the method returns `fail` and the step size dt is reduced. If the error estimate is acceptably small, the step is performed, success is returned and dt might be increased to make the steps as large as possible. This method also updates t if a step is performed.
   
   Parameters:  
   - `dt` The step size. Updated.  
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
   - `t` The value of the time. Updated if the step is successful.
The state of the ODE which should be solved. Overwritten if the step is successful. Can be a boost range.

Returns: success if the step was accepted, fail otherwise.

3. template<typename System, typename StateInOut, typename DerivIn>
   controlled_step_result
   try_step(System system, StateInOut & x, const DerivIn & dxdt, time_type & t,
            time_type & dt);

Tries to perform one step.

This method tries to do one step with step size dt. If the error estimate is too large, the step is rejected and the method returns fail and the step size dt is reduced. If the error estimate is acceptably small, the step is performed, success is returned and dt might be increased to make the steps as large as possible. This method also updates t if a step is performed.

Parameters: dt The step size. Updated.
             dxdt The derivative of state.
             system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
             t The value of the time. Updated if the step is successful.
             x The state of the ODE which should be solved. Overwritten if the step is successful.

Returns: success if the step was accepted, fail otherwise.

4. template<typename System, typename StateIn, typename StateOut>
   boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type
   try_step(System system, const StateIn & in, time_type & t, StateOut & out,
             time_type & dt);

Tries to perform one step.

Note
This method is disabled if state_type=time_type to avoid ambiguity.

This method tries to do one step with step size dt. If the error estimate is too large, the step is rejected and the method returns fail and the step size dt is reduced. If the error estimate is acceptably small, the step is performed, success is returned and dt might be increased to make the steps as large as possible. This method also updates t if a step is performed.

Parameters: dt The step size. Updated.
             in The state of the ODE which should be solved.
             out Used to store the result of the step.
             system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
             t The value of the time. Updated if the step is successful.

Returns: success if the step was accepted, fail otherwise.

5. template<typename System, typename StateIn, typename DerivIn,
           typename StateOut>
   controlled_step_result
   try_step(System system, const StateIn & in, const DerivIn & dxdt,
             time_type & t, StateOut & out, time_type & dt);

Tries to perform one step.

This method tries to do one step with step size dt. If the error estimate is too large, the step is rejected and the method returns fail and the step size dt is reduced. If the error estimate is acceptably small, the step is performed, success is returned and dt might be increased to make the steps as large as possible. This method also updates t if a step is performed.

Parameters: dt The step size. Updated.
dxdt  The derivative of state.
in   The state of the ODE which should be solved.
out  Used to store the result of the step.
system  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
t  The value of the time. Updated if the step is successful.

Returns:

6. `value_type last_error(void) const;`

Returns the error of the last step.

returns The last error of the step.

7. `template<typename StateType> void adjust_size(const StateType & x);`

Adjust the size of all temporaries in the stepper manually.

Parameters:  
  `x`  A state from which the size of the temporaries to be resized is deduced.

8. `stepper_type & stepper(void);`

Returns the instance of the underlying stepper.

Returns:  
The instance of the underlying stepper.

9. `const stepper_type & stepper(void) const;`

Returns the instance of the underlying stepper.

Returns:  
The instance of the underlying stepper.

`controlled_runge_kutta` private member functions

1. `template<typename System, typename StateInOut> 
   controlled_step_result
   try_step_v1(System system, StateInOut & x, time_type & t, time_type & dt);`

2. `template<typename StateIn> bool resize_m_xerr_impl(const StateIn & x);`

3. `template<typename StateIn> bool resize_m_dxdt_impl(const StateIn & x);`

4. `template<typename StateIn> bool resize_m_xnew_impl(const StateIn & x);`

Class template `controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, 
explicit_error_stepper_fsal_tag>`

boost::numeric::odeint::controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, 
explicit_error_stepper_fsal_tag> — Implements step size control for Runge-Kutta FSAL steppers with error estimation.
Synopsis

// In header: <boost/numeric/odeint/stepper/controlled_runge_kutta.hpp>

template<typename ErrorStepper, typename ErrorChecker, typename Resizer>
class controlled_runge_kutta<ErrorStepper, ErrorChecker, Resizer, explicit_error_step_per_fsal_tag> {
public:
    // types
    typedef ErrorStepper stepper_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::time_type time_type;
    typedef stepper_type::algebra_type algebra_type;
    typedef stepper_type::operations_type operations_type;
    typedef Resizer resizer_type;
    typedef ErrorChecker error_checker_type;
    typedef explicit_controlled_stepper_fsal_tag stepper_category;

    // construct/copy/destruct
    controlled_runge_kutta(const error_checker_type & = error_checker_type(),
                        const stepper_type & = stepper_type());

    // public member functions
    template<typename System, typename StateInOut>
    controlled_step_result try_step(System, StateInOut &, time_type &, time_type &);
    template<typename System, typename StateInOut>
    controlled_step_result try_step(System, const StateInOut &, time_type &, time_type &);
    template<typename System, typename StateIn, typename StateOut>
    boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type
    try_step(System, const StateIn &, time_type &, StateOut &, time_type &);
    template<typename System, typename StateInOut, typename DerivInOut>
    controlled_step_result try_step(System, StateInOut &, DerivInOut &, time_type &, time_type &);
    template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename DerivOut>
    controlled_step_result try_step(System, const StateIn &, const DerivIn &, time_type &, StateOut &, DerivOut &, time_type &);
    void reset(void);
    template<typename DerivIn> void initialize(const DerivIn &);
    template<typename System, typename StateIn>
    void initialize(System, const StateIn &, time_type);
    bool is_initialized(void) const;
    template<typename StateType> void adjust_size(const StateType &);
    stepper_type & stepper(void);
    const stepper_type & stepper(void) const;

    // private member functions
    template<typename StateIn>
    bool resize_m_xerr_impl(const StateIn &);
    template<typename StateIn>
    bool resize_m_dxdt_impl(const StateIn &);
    template<typename StateIn>
    bool resize_m_dxdt_new_impl(const StateIn &);
    template<typename StateIn>
    bool resize_m_xnew_impl(const StateIn &);
    template<typename System, typename StateInOut>
    controlled_step_result try_step_v1(System, StateInOut &, time_type &, time_type &);
};
Description

This class implements the step size control for FSAL Runge-Kutta steppers with error estimation.

Template Parameters

1. `typename ErrorStepper`
   The stepper type with error estimation, has to fulfill the ErrorStepper concept.

2. `typename ErrorChecker`
   The error checker

3. `typename Resizer`
   The resizer policy type.

**controlled_runge_kutta** public construct/copy/destroy

1. ```cpp
controlled_runge_kutta(const error_checker_type & error_checker = error_checker_type(), const stepper_type & stepper = stepper_type());
```

Constructs the controlled Runge-Kutta stepper.

Parameters:
- `error_checker` An instance of the error checker.
- `stepper` An instance of the underlying stepper.

**controlled_runge_kutta** public member functions

1. ```cpp
template<typename System, typename StateInOut>
controlled_step_result try_step(System system, StateInOut & x, time_type & t, time_type & dt);
```

Tries to perform one step.

This method tries to do one step with step size `dt`. If the error estimate is too large, the step is rejected and the method returns fail and the step size `dt` is reduced. If the error estimate is acceptably small, the step is performed, success is returned and `dt` might be increased to make the steps as large as possible. This method also updates `t` if a step is performed.

Parameters:
- `dt` The step size. Updated.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time. Updated if the step is successful.
- `x` The state of the ODE which should be solved. Overwritten if the step is successful.

Returns: success if the step was accepted, fail otherwise.

2. ```cpp
template<typename System, typename StateInOut>
controlled_step_result try_step(System system, const StateInOut & x, time_type & t, time_type & dt);
```

Tries to perform one step. Solves the forwarding problem and allows for using boost range as state_type.

This method tries to do one step with step size `dt`. If the error estimate is too large, the step is rejected and the method returns fail and the step size `dt` is reduced. If the error estimate is acceptably small, the step is performed, success is returned and `dt` might be increased to make the steps as large as possible. This method also updates `t` if a step is performed.
Parameters:  
- \( \text{dt} \): The step size. Updated.
- \( \text{system} \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.
- \( x \): The state of the ODE which should be solved. Overwritten if the step is successful. Can be a boost range.

Returns: success if the step was accepted, fail otherwise.

3. `template<typename System, typename StateIn, typename StateOut>`

   ```cpp
   boost::disable_if< boost::is_same< StateIn, time_type >, controlled_step_result >::type
   try_step(System system, const StateIn & in, time_type & t, StateOut & out, time_type & dt);
   ```

Tries to perform one step.

**Note**

This method is disabled if state_type=time_type to avoid ambiguity.

This method tries to do one step with step size \( dt \). If the error estimate is too large, the step is rejected and the method returns fail and the step size \( dt \) is reduced. If the error estimate is acceptably small, the step is performed, success is returned and \( dt \) might be increased to make the steps as large as possible. This method also updates \( t \) if a step is performed.

Parameters:  
- \( \text{dt} \): The step size. Updated.
- \( \text{in} \): The state of the ODE which should be solved.
- \( \text{out} \): Used to store the result of the step.
- \( \text{system} \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.

Returns: success if the step was accepted, fail otherwise.

4. `template<typename System, typename StateInOut, typename DerivInOut>`

   ```cpp
   controlled_step_result
   try_step(System system, StateInOut & x, DerivInOut & dxdt, time_type & t, time_type & dt);
   ```

Tries to perform one step.

This method tries to do one step with step size \( dt \). If the error estimate is too large, the step is rejected and the method returns fail and the step size \( dt \) is reduced. If the error estimate is acceptably small, the step is performed, success is returned and \( dt \) might be increased to make the steps as large as possible. This method also updates \( t \) if a step is performed.

Parameters:  
- \( \text{dt} \): The step size. Updated.
- \( \text{dxdt} \): The derivative of state.
- \( \text{system} \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \): The value of the time. Updated if the step is successful.
- \( x \): The state of the ODE which should be solved. Overwritten if the step is successful.

Returns: success if the step was accepted, fail otherwise.

5. `template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename DerivOut>`

   ```cpp
   controlled_step_result
   try_step(System system, const StateIn & in, const DerivIn & dxdt_in, time_type & t, StateOut & out, DerivOut & dxdt_out, time_type & dt);
   ```

Tries to perform one step.
This method tries to do one step with step size $dt$. If the error estimate is too large, the step is rejected and the method returns fail and the step size $dt$ is reduced. If the error estimate is acceptably small, the step is performed, success is returned and $dt$ might be increased to make the steps as large as possible. This method also updates $t$ if a step is performed.

Parameters:
- $dt$  The step size. Updated.
- $in$  The state of the ODE which should be solved.
- $out$  Used to store the result of the step.
- $system$  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- $t$  The value of the time. Updated if the step is successful.

Returns:
- success if the step was accepted, fail otherwise.

6. `void reset(void);`

Resets the internal state of the underlying FSAL stepper.

7. `template<typename DerivIn> void initialize(const DerivIn & deriv);`

Initializes the internal state storing an internal copy of the derivative.

Parameters:
- $deriv$  The initial derivative of the ODE.

8. `template<typename System, typename StateIn> void initialize(System system, const StateIn & x, time_type t);`

Initializes the internal state storing an internal copy of the derivative.

Parameters:
- $system$  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- $t$  The initial time.
- $x$  The initial state of the ODE which should be solved.

9. `bool is_initialized(void) const;`

Returns true if the stepper has been initialized, false otherwise.

Returns:
- true, if the stepper has been initialized, false otherwise.

10. `template<typename StateType> void adjust_size(const StateType & x);`

Adjust the size of all temporaries in the stepper manually.

Parameters:
- $x$  A state from which the size of the temporaries to be resized is deduced.

11. `stepper_type & stepper(void);`

Returns the instance of the underlying stepper.

Returns:
- The instance of the underlying stepper.

12. `const stepper_type & stepper(void) const;`

Returns the instance of the underlying stepper.

Returns:
- The instance of the underlying stepper.
controlled_runge_kutta private member functions

1. template<typename StateIn> bool resize_m_xerr_impl(const StateIn & x);

2. template<typename StateIn> bool resize_m_dxdt_impl(const StateIn & x);

3. template<typename StateIn> bool resize_m_dxdt_new_impl(const StateIn & x);

4. template<typename StateIn> bool resize_m_xnew_impl(const StateIn & x);

5. template<typename System, typename StateInOut>
controlled_step_result try_step_v1(System system, StateInOut & x, time_type & t, time_type & dt);

Header `<boost/numeric/odeint/stepper/controlled_step_result.hpp>`

namespace boost {
    namespace numeric {
        namespace odeint {
            // Enum representing the return values of the controlled steppers.
            enum controlled_step_result { success, fail }
        }
    }
}

Header `<boost/numeric/odeint/stepper/dense_output_runge_kutta.hpp>`

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper,
                typename StepperCategory = typename Stepper::stepper_category>
            class dense_output_runge_kutta;

            template<typename Stepper>
            class dense_output_runge_kutta<Stepper, stepper_tag>;
            template<typename Stepper>
            class dense_output_runge_kutta<Stepper, explicit_controlled_stepper_fsal_tag>;
        }
    }
}

Class template dense_output_runge_kutta

boost::numeric::odeint::dense_output_runge_kutta
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/dense_output_runge_kutta.hpp>

template<typename Stepper,
         typename StepperCategory = typename Stepper::stepper_category>
class dense_output_runge_kutta {
};
```

Description

Specializations

- Class template `dense_output_runge_kutta<Stepper, stepper_tag>`
- Class template `dense_output_runge_kutta<Stepper, explicit_controlled_stepper_fsal_tag>`

**Class template dense_output_runge_kutta<Stepper, stepper_tag>**

`boost::numeric::odeint::dense_output_runge_kutta<Stepper, stepper_tag>` — The class representing dense-output Runge-Kutta steppers.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/dense_output_runge_kutta.hpp>

template<typename Stepper>
class dense_output_runge_kutta<Stepper, stepper_tag> { 
public:
    // types
    typedef Stepper stepper_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::wrapped_state_type wrapped_state_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::wrapped_deriv_type wrapped_deriv_type;
    typedef stepper_type::time_type time_type;
    typedef stepper_type::algebra_type algebra_type;
    typedef stepper_type::operations_type operations_type;
    typedef stepper_type::resizer_type resizer_type;
    typedef dense_output_stepper_tag stepper_category;
    typedef dense_output_runge_kutta< Stepper > dense_output_stepper_type;

    // construct/copy/destruct
    dense_output_runge_kutta(const stepper_type & = stepper_type());

    // public member functions
    template<typename StateType>
    void initialize(const StateType &state, time_type, time_type);
    template<typename System> std::pair< time_type, time_type > do_step(System);
    template<typename StateOut> void calc_state(time_type, StateOut &state) const;
    template<typename StateOut>
    void calc_state(time_type, const StateOut &state) const;
    template<typename StateType> void adjust_size(const StateType &state);
    const state_type & current_state(void) const;
    time_type current_time(void) const;
    const state_type & previous_state(void) const;
    time_type previous_time(void) const;

    // private member functions
    state_type & get_current_state(void);
    const state_type & get_current_state(void) const;
    state_type & get_old_state(void);
    const state_type & get_old_state(void) const;
    void toggle_current_state(void);
    template<typename StateIn>
    bool resize_impl(const StateIn &state);
};
```

Description

Note

In this stepper, the initialize method has to be called before using the do_step method.

The dense-output functionality allows to interpolate the solution between subsequent integration points using intermediate results obtained during the computation. This version works based on a normal stepper without step-size control.

Template Parameters

1. **typename Stepper**
The stepper type of the underlying algorithm.

**dense_output_runge_kutta public construct/copy/destruct**

1. ```cpp
   dense_output_runge_kutta(const stepper_type & stepper = stepper_type());
```

Constructs the dense_output_runge_kutta class. An instance of the underlying stepper can be provided.

Parameters: `stepper` An instance of the underlying stepper.

**dense_output_runge_kutta public member functions**

1. ```cpp
   template<typename StateType>
   void initialize(const StateType & x0, time_type t0, time_type dt0);
```

Initializes the stepper. Has to be called before do_step can be used to set the initial conditions and the step size.

Parameters: `dt0` The step size.  
`t0` The initial time, at which the step should be performed.  
`x0` The initial state of the ODE which should be solved.

2. ```cpp
   template<typename System>
   std::pair< time_type, time_type > do_step(System system);
```

Does one time step.

**Note**

initialize has to be called before using this method to set the initial conditions x,t and the step size.

Parameters: `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.

Returns: Pair with start and end time of the integration step.

3. ```cpp
   template<typename StateOut> void calc_state(time_type t, StateOut & x) const;
```

Calculates the solution at an intermediate point.

Parameters: `t` The time at which the solution should be calculated, has to be in the current time interval.  
`x` The output variable where the result is written into.

4. ```cpp
   template<typename StateOut>
   void calc_state(time_type t, const StateOut & x) const;
```

Calculates the solution at an intermediate point. Solves the forwarding problem.

Parameters: `t` The time at which the solution should be calculated, has to be in the current time interval.  
`x` The output variable where the result is written into, can be a boost range.

5. ```cpp
   template<typename StateType> void adjust_size(const StateType & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters: `x` A state from which the size of the temporaries to be resized is deduced.
6. `const state_type & current_state(void) const;`

Returns the current state of the solution.
Returns: The current state of the solution $x(t)$.

7. `time_type current_time(void) const;`

Returns the current time of the solution.
Returns: The current time of the solution $t$.

8. `const state_type & previous_state(void) const;`

Returns the last state of the solution.
Returns: The last state of the solution $x(t-dt)$.

9. `time_type previous_time(void) const;`

Returns the last time of the solution.
Returns: The last time of the solution $t-dt$.

**dense_output_runge_kutta** private member functions

1. `state_type & get_current_state(void);`

2. `const state_type & get_current_state(void) const;`

3. `state_type & get_old_state(void);`

4. `const state_type & get_old_state(void) const;`

5. `void toggle_current_state(void);`

6. `template<typename StateIn> bool resize_impl(const StateIn & x);`

**Class template** `dense_output_runge_kutta<Stepper, explicit_controlled_stepper_fsal_tag>`

`boost::numeric::odeint::dense_output_runge_kutta<Stepper, explicit_controlled_stepper_fsal_tag>` — The class representing dense-output Runge-Kutta steppers with FSAL property.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/dense_output_runge_kutta.hpp>

template<typename Stepper>
class dense_output_runge_kutta<Stepper, explicit_controlled_stepper_fsal_tag> {
public:
    // types
    typedef Stepper controlled_stepper_type;
    typedef controlled_stepper_type::stepper_type stepper_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::wrapped_state_type wrapped_state_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::wrapped_deriv_type wrapped_deriv_type;
    typedef stepper_type::time_type time_type;
    typedef stepper_type::algebra_type algebra_type;
    typedef stepper_type::operations_type operations_type;
    typedef stepper_type::resizer_type resizer_type;
    typedef dense_output_runge_kutta< Stepper > stepper_category;
    typedef dense_output_runge_kutta< Stepper > dense_output_stepper_type;

    // construct/copy/destruct
    dense_output_runge_kutta(const controlled_stepper_type & = controlled_stepper_type());

    // public member functions
    template<typename StateType>
    void initialize(const StateType &, time_type, time_type);
    template<typename System> std::pair< time_type, time_type > do_step(System);
    template<typename StateOut> void calc_state(time_type, StateOut &) const;
    template<typename StateOut>
    void calc_state(time_type, const StateOut &) const;
    template<typename StateIn> bool resize(const StateIn &);
    template<typename StateType> void adjust_size(const StateType &);
    const state_type & current_state(void) const;
    time_type current_time(void) const;
    const state_type & previous_state(void) const;
    time_type previous_time(void) const;
    time_type current_time_step(void) const;

    // private member functions
    state_type & get_current_state(void);
    const state_type & get_current_state(void) const;
    state_type & get_old_state(void);
    const state_type & get_old_state(void) const;
    deriv_type & get_current_deriv(void);
    const deriv_type & get_current_deriv(void) const;
    deriv_type & get_old_deriv(void);
    const deriv_type & get_old_deriv(void) const;
    void toggle_current_state(void);
};
```

Description

The interface is the same as for `dense_output_runge_kutta< Stepper, stepper_tag >`. This class provides dense output functionality based on methods with step size controlled.

Template Parameters

1. `typename Stepper`
The stepper type of the underlying algorithm.

**dense_output_runge_kutta**

**public construct/copy/destruct**

1. `dense_output_runge_kutta(const controlled_stepper_type & stepper = controlled_stepper_type());`

**dense_output_runge_kutta**

**public member functions**

1. `template<typename StateType> void initialize(const StateType & x0, time_type t0, time_type dt0);`

2. `template<typename System> std::pair< time_type, time_type > do_step(System system);`

3. `template<typename StateOut> void calc_state(time_type t, StateOut & x) const;`

4. `template<typename StateOut> void calc_state(time_type t, const StateOut & x) const;`

5. `template<typename StateIn> bool resize(const StateIn & x);`

6. `template<typename StateType> void adjust_size(const StateType & x);`

7. `const state_type & current_state(void) const;`

8. `time_type current_time(void) const;`

9. `const state_type & previous_state(void) const;`

10. `time_type previous_time(void) const;`

11. `time_type current_time_step(void) const;`

**dense_output_runge_kutta**

**private member functions**

1. `state_type & get_current_state(void);`

2. `const state_type & get_current_state(void) const;`
3. `state_type & get_old_state(void);`

4. `const state_type & get_old_state(void) const;`

5. `deriv_type & get_current_deriv(void);`

6. `const deriv_type & get_current_deriv(void) const;`

7. `deriv_type & get_old_deriv(void);`

8. `const deriv_type & get_old_deriv(void) const;`

9. `void toggle_current_state(void);`

**Header `<boost/numeric/odeint/stepper/euler.hpp>`**

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                     typename Deriv = State, typename Time = Value,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
            class euler;
        }
    }
}
```

**Class template euler**

`boost::numeric::odeint::euler` — An implementation of the Euler method.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/euler.hpp>

template<typename State, typename Value = double, typename Deriv = State,
    typename Time = Value, typename Algebra = range_algebra,
    typename Operations = default_operations,
    typename Resizer = initially_resizer>
class euler : public boost::numeric::odeint::explicit_stepper_base< Stepper, Order, State, Value,
    Deriv, Time, Algebra, Operations, Resizer >
{
    public:
        // types
        typedef explicit_stepper_base< euler< ... >,... > stepper_base_type;
        typedef stepper_base_type::state_type state_type;
        typedef stepper_base_type::value_type value_type;
        typedef stepper_base_type::deriv_type deriv_type;
        typedef stepper_base_type::time_type time_type;
        typedef stepper_base_type::algebra_type algebra_type;
        typedef stepper_base_type::operations_type operations_type;
        typedef stepper_base_type::resizer_type resizer_type;

        // construct/copy/destroy
        euler(const algebra_type & = algebra_type());

        // public member functions
        template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
            void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                StateOut &, time_type);
        template<typename StateOut, typename StateIn1, typename StateIn2>
            void calc_state(StateOut &, time_type, const StateIn1 &, time_type,
                const StateIn2 &, time_type) const;
        template<typename StateType> void adjust_size(const StateType &);
        order_type order(void) const;
        template<typename System, typename StateInOut>
            void do_step(System, StateInOut &, time_type, time_type);
        template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
            void do_step(System, const StateIn &, const DerivIn &, time_type,
                StateOut &, time_type);
        template<typename StateIn> void adjust_size(const StateIn &);
        algebra_type & algebra();
        const algebra_type & algebra() const;
};
```

Description

The Euler method is a very simply solver for ordinary differential equations. This method should not be used for real applications. It is only useful for demonstration purposes. Step size control is not provided but trivial continuous output is available.

This class derives from `explicit_stepper_base` and inherits its interface via CRTP (current recurring template pattern), see `explicit_stepper_base`
Template Parameters

1. **typename State**
   The state type.

2. **typename Value = double**
   The value type.

3. **typename Deriv = State**
   The type representing the time derivative of the state.

4. **typename Time = Value**
   The time representing the independent variable - the time.

5. **typename Algebra = range_algebra**
   The algebra type.

6. **typename Operations = default_operations**
   The operations type.

7. **typename Resizer = initially_resizer**
   The resizer policy type.

euler public construct/copy/destruct

1. `euler(const algebra_type & algebra = algebra_type());`
   Constructs the euler class. This constructor can be used as a default constructor of the algebra has a default constructor.

Parameters:   
   algebra A copy of algebra is made and stored inside `explicit_stepper_base`.

euler public member functions

1. ```
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                      time_type t, StateOut & out, time_type dt);
```  
   This method performs one step. The derivative \(dxdt\) of \(in\) at the time \(t\) is passed to the method. The result is updated out of place, hence the input is in \(in\) and the output in \(out\). Access to this step functionality is provided by `explicit_stepper_base` and `do_step_impl` should not be called directly.

Parameters:    
   dt The step size.
   dxdt The derivative of \(x\) at \(t\).
   in The state of the ODE which should be solved. \(in\) is not modified in this method
   out The result of the step is written in \(out\).
The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.

The value of the time, at which the step should be performed.

```cpp
template<typename StateOut, typename StateIn1, typename StateIn2>
void calc_state(StateOut & x, time_type t, const StateIn1 & old_state,
    time_type t_old, const StateIn2 & current_state,
    time_type t_new) const;
```

This method is used for continuous output and it calculates the state \( x \) at a time \( t \) from the knowledge of two states \( \text{old}_\text{state} \) and \( \text{current}_\text{state} \) at time points \( t\_\text{old} \) and \( t\_\text{new} \).

```cpp
template<typename StateType> void adjust_size(const StateType & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters:
- \( x \): A state from which the size of the temporaries to be resized is deduced.

Returns:
- Returns the order of the stepper.

```cpp
template<typename System, typename StateInOut>
void do_step(System system, StateInOut & x, time_type t, time_type dt);
```

This method performs one step. It transforms the result in-place.

Parameters:
- \( dt \): The step size.
- \( system \): The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- \( t \): The value of the time, at which the step should be performed.
- \( x \): The state of the ODE which should be solved. After calling do_step the result is updated in \( x \).

```cpp
template<typename System, typename StateInOut, typename DerivIn>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
    time_type dt);
```

The method performs one step. Additionally to the other method the derivative of \( x \) is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in \( x \). This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

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Parameters: 

- **dt**  The step size.
- **dxdt**  The derivative of x at t.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.
- **x**  The state of the ODE which should be solved. After calling do_step the result is updated in x.

8. 

```cpp
template<typename System, typename StateIn, typename StateOut>
void do_step(System system, const StateIn & in, time_type t, StateOut & out,
             time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place.

**Note**

This method does not solve the forwarding problem.

Parameters: 

- **dt**  The step size.
- **in**  The state of the ODE which should be solved. in is not modified in this method
- **out**  The result of the step is written in out.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.

9. 

```cpp
template<typename System, typename StateIn, typename DerivIn,
         typename StateOut>
void do_step(System system, const StateIn & in, const DerivIn & dxdt,
             time_type t, StateOut & out, time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

**Note**

This method does not solve the forwarding problem.

Parameters: 

- **dt**  The step size.
- **dxdt**  The derivative of x at t.
- **in**  The state of the ODE which should be solved. in is not modified in this method
- **out**  The result of the step is written in out.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.

10. 

```cpp
template<typename StateIn>
void adjust_size(const StateIn & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters: 

- **x**  A state from which the size of the temporaries to be resized is deduced.

11. 

```cpp
algebra_type & algebra();
```

Returns:  A reference to the algebra which is held by this class.
const algebra_type & algebra() const;

Returns: A const reference to the algebra which is held by this class.

Header `<boost/numeric/odeint/stepper/explicit_error_generic_rk.hpp>`

class explicit_error_generic_rk

boost::numeric::odeint::explicit_error_generic_rk — A generic implementation of explicit Runge-Kutta algorithms with error estimation. This class is as a base class for all explicit Runge-Kutta steppers with error estimation.
Synopsis

// In header: <boost/numeric/odeint/stepper/explicit_error_generic_rk.hpp>

template<typename StageCount, size_t Order, size_t StepperOrder,
    size_t ErrorOrder, typename State, typename Value = double,
    typename Deriv = State, typename Time = Value,
    typename Algebra = range_algebra,
    typename Operations = default_operations,
    typename Resizer = initially_resizer>

class explicit_error_generic_rk : public boost::numeric::odeint::explicit_error_stepper_base<>
{
    public:
     // types
     typedef explicit_stepper_base< ... > stepper_base_type;
     typedef stepper_base_type::state_type state_type;
     typedef stepper_base_type::wrapped_state_type wrapped_state_type;
     typedef stepper_base_type::value_type value_type;
     typedef stepper_base_type::deriv_type deriv_type;
     typedef stepper_base_type::wrapped_deriv_type wrapped_deriv_type;
     typedef stepper_base_type::time_type time_type;
     typedef stepper_base_type::algebra_type algebra_type;
     typedef stepper_base_type::operations_type operations_type;
     typedef stepper_base_type::resizer_type resizer_type;
     typedef unspecified rk_algorithm_type;
     typedef rk_algorithm_type::coef_a_type coef_a_type;
     typedef rk_algorithm_type::coef_b_type coef_b_type;
     typedef rk_algorithm_type::coef_c_type coef_c_type;

     // construct/copy/destruct
     explicit_error_generic_rk(const coef_a_type &, const coef_b_type &,
                  const coef_b_type &, const coef_c_type &, const algebra_type & = algebra_type());

     // public member functions
     template<typename System, typename StateIn, typename DerivIn,
                typename StateOut, typename Err>
        void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                           StateOut &, time_type, Err &);

     template<typename System, typename StateIn, typename DerivIn,
                typename StateOut>
        void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                           StateOut &, time_type);

     template<typename StateIn> void adjust_size(const StateIn &);

     order_type order(void) const;

     order_type stepper_order(void) const;

     order_type error_order(void) const;

     template<typename System, typename StateInOut>
        void do_step(System, StateInOut &, time_type, time_type);

     template<typename System, typename StateIn, typename StateOut>
        void do_step(System, const StateIn &, time_type, StateOut &, time_type);

     template<typename System, typename StateIn, typename DerivIn>
        void do_step(System, const StateIn &, const DerivIn &, time_type, time_type);

     boost::disable_if<boost::is_same< DerivIn, time_type >, void>::type
do_step(System, StateIn &, const DerivIn &, time_type, time_type);

     boost::disable_if<boost::is_same< StateIn, time_type >, void>::type
do_step(System, const StateIn &, time_type, StateOut &, time_type);

     boost::disable_if<boost::is_same< DerivIn, time_type >, void>::type
do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type);

template<typename System, typename StateIn, typename DerivIn, typename Err>
void do_step(System, StateIn &, time_type, time_type, Err &);

template<typename System, typename StateIn, typename StateOut, typename DerivIn, typename Err>
void do_step(System, const StateIn &, time_type, time_type, StateOut &, time_type, Err &);

boost::disable_if<boost::is_same< DerivIn, time_type >, void >::type
do_step(System, StateIn &, const DerivIn &, time_type, time_type, StateOut &, time_type, Err &);

template<typename System, typename StateIn, typename StateOut, typename DerivIn, typename StateOut, typename Err>
do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type, time_type, Err &);

algebra_type & algebra();
const algebra_type & algebra() const;

// private member functions
template<typename StateIn> bool resize_impl(const StateIn &);

// public data members
static const size_t stage_count;

Description

This class implements the explicit Runge-Kutta algorithms with error estimation in a generic way. The Butcher tableau is passed to the stepper which constructs the stepper scheme with the help of a template-metaprogramming algorithm. ToDo: Add example!

This class derives explicit_error_stepper_base which provides the stepper interface.

Template Parameters

1. size_t StageCount

The number of stages of the Runge-Kutta algorithm.

2. size_t Order

The order of a stepper if the stepper is used without error estimation.

3. size_t StepperOrder

The order of a step if the stepper is used with error estimation. Usually Order and StepperOrder have the same value.

4. size_t ErrorOrder

The order of the error step if the stepper is used with error estimation.

5. typename State

The type representing the state of the ODE.
6. \texttt{typename Value = double}

The floating point type which is used in the computations.

7. \texttt{typename Deriv = State}

8. \texttt{typename Time = Value}

The type representing the independent variable - the time - of the ODE.

9. \texttt{typename Algebra = range\_algebra}

The algebra type.

10. \texttt{typename Operations = default\_operations}

The operations type.

11. \texttt{typename Resizer = initially\_resizer}

The resizer policy type.

\texttt{explicit\_error\_generic\_rk} \texttt{public construct/copy/destruct}

1. \texttt{explicit\_error\_generic\_rk(const coef\_a\_type & a, const coef\_b\_type & b, const coef\_b\_type & b2, const coef\_c\_type & c, const algebra\_type & algebra = algebra\_type());}

Constructs the \texttt{explicit\_error\_generik\_rk} class with the given parameters \(a, b, b2\) and \(c\). See examples section for details on the coefficients.

Parameters:
\begin{itemize}
  \item \texttt{a} Triangular matrix of parameters \(b\) in the Butcher tableau.
  \item \texttt{algebra} A copy of algebra is made and stored inside \texttt{explicit\_stepper\_base}.
  \item \texttt{b} Last row of the butcher tableau.
  \item \texttt{b2} Parameters for lower-order evaluation to estimate the error.
  \item \texttt{c} Parameters to calculate the time points in the Butcher tableau.
\end{itemize}

\texttt{explicit\_error\_generic\_rk} \texttt{public member functions}

1. \texttt{template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err> void do\_step\_impl(System system, const StateIn & in, const DerivIn & dxdt, time\_type t, StateOut & out, time\_type dt, Err & xerr);}

This method performs one step. The derivative \(dxdt\) of \(in\) at the time \(t\) is passed to the method. The result is updated out-of-place, hence the input is \(in\) and the output in \(out\). Futhermore, an estimation of the error is stored in \(xerr\). \texttt{do\_step\_impl} is used by \texttt{explicit\_error\_stepper\_base}.

Parameters:
\begin{itemize}
  \item \texttt{dt} The step size.
  \item \texttt{dxdt} The derivative of \(x\) at \(t\).
  \item \texttt{in} The state of the ODE which should be solved. \(in\) is not modified in this method
  \item \texttt{out} The result of the step is written in \(out\).
  \item \texttt{system} The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
\end{itemize}
t      The value of the time, at which the step should be performed.

xerr   The result of the error estimation is written in xerr.

2. template<typename System, typename StateIn, typename DerivIn, typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                      time_type t, StateOut & out, time_type dt);

This method performs one step. The derivative dxdt of in at the time t is passed to the method. The result is updated out-of-place, hence the input is in and the output in out. Access to this step functionality is provided by explicit_stepper_base and do_step_impl should not be called directly.

Parameters:
- dt      The step size.
- dxdt    The derivative of x at t.
- in      The state of the ODE which should be solved. in is not modified in this method
- out     The result of the step is written in out.
- system  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- t       The value of the time, at which the step should be performed.

3. template<typename StateIn> void adjust_size(const StateIn & x);

Adjust the size of all temporaries in the stepper manually.

Parameters:  x  A state from which the size of the temporaries to be resized is deduced.

4. order_type order(void) const;

Returns:     Returns the order of the stepper if it used without error estimation.

5. order_type stepper_order(void) const;

Returns:     Returns the order of a step if the stepper is used without error estimation.

6. order_type error_order(void) const;

Returns:     Returns the order of an error step if the stepper is used without error estimation.

7. template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);

This method performs one step. It transforms the result in-place.

Parameters:
- dt      The step size.
- system  The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- t       The value of the time, at which the step should be performed.
- x       The state of the ODE which should be solved. After calling do_step the result is updated in x.

8. template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.
9. template<typename System, typename StateInOut, typename DerivIn>
 boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
          time_type dt);

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**
This method does not solve the forwarding problem.

**Parameters:**
- dt: The step size.
- dxdt: The derivative of x at t.
- system: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- t: The value of the time, at which the step should be performed.
- x: The state of the ODE which should be solved. After calling do_step the result is updated in x.

10. template<typename System, typename StateIn, typename StateOut>
 boost::disable_if< boost::is_same< StateIn, time_type >, void >::type
do_step(System system, const StateIn & in, time_type t, StateOut & out,
         time_type dt);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other do_step variants.

**Note**
This method does not solve the forwarding problem.

**Parameters:**
- dt: The step size.
- in: The state of the ODE which should be solved. in is not modified in this method
- out: The result of the step is written in out.
- system: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- t: The value of the time, at which the step should be performed.

11. template<typename System, typename StateIn, typename DerivIn,
         typename StateOut>
 boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, const StateIn & in, const DerivIn & dxdt,
         time_type t, StateOut & out, time_type dt);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```
This method is disabled if DerivIn and Time are of same type.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

```cpp
template<typename System, typename StateInOut, typename Err>
void do_step(System system, StateInOut & x, time_type t, time_type dt,
             Err & xerr);
```

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

Parameters:
- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. x is updated by this method.
- **xerr**: The estimation of the error is stored in xerr.

```cpp
template<typename System, typename StateInOut, typename Err>
void do_step(System system, const StateInOut & x, time_type t, time_type dt,
             Err & xerr);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

```cpp
template<typename System, typename StateInOut, typename DerivIn, typename Err>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt , xerr );
```

The result is updated in place in x. This method is disabled if Time and DerivIn are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
The state of the ODE which should be solved. After calling do_step the result is updated in x.
xerr
The error estimate is stored in xerr.

```cpp
template<typename System, typename StateIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, time_type t, StateOut & out,
time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

```cpp
template<typename System, typename StateIn, typename DerivIn,
typeiname StateOut, typename Err>
void do_step(System system, const StateIn & in, const DerivIn & dxdt,
time_type t, StateOut & out, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if DerivIn and Time are of same type.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

```cpp
algebra_type & algebra();
```

Returns: A reference to the algebra which is held by this class.

```cpp
const algebra_type & algebra() const;
```

Returns: A const reference to the algebra which is held by this class.
explicit_error_generic_rk private member functions

1. template<typename StateIn> bool resize_impl(const StateIn & x);

Header `<boost/numeric/odeint/stepper/explicit_generic_rk.hpp>`

```
namespace boost {
    namespace numeric {
        namespace odeint {
            template<size_t StageCount, size_t Order, typename State,
            typename Value, typename Deriv, typename Time,
            typename Algebra, typename Operations, typename Resizer>
                class explicit_generic_rk;
        }
    }
}
```

Class template explicit_generic_rk

boost::numeric::odeint::explicit_generic_rk — A generic implementation of explicit Runge-Kutta algorithms. This class is as a base class for all explicit Runge-Kutta steppers.
Synopsis

```cpp
// In header: &lt;boost/numeric/odeint/stepper/explicit_generic_rk.hpp&gt;

template&lt;size_t StageCount, size_t Order, typename State, typename Value,
          typename Deriv, typename Time, typename Algebra, typename Operations,
          typename Resizer&gt;
class explicit_generic_rk : public boost::numeric::odeint::explicit_stepper_base&lt; Stepper, Order,
true, State, Value, Deriv, Time, Algebra, Operations, Resizer &gt;
{
public:
  // types
  typedef explicit_stepper_base&lt; ... &gt; stepper_base_type;
  typedef stepper_base_type::state_type state_type;
  typedef stepper_base_type::wrapped_state_type wrapped_state_type;
  typedef stepper_base_type::value_type value_type;
  typedef stepper_base_type::deriv_type deriv_type;
  typedef stepper_base_type::wrapped_deriv_type wrapped_deriv_type;
  typedef stepper_base_type::time_type time_type;
  typedef stepper_base_type::algebra_type algebra_type;
  typedef stepper_base_type::operations_type operations_type;
  typedef stepper_base_type::resizer_type resizer_type;
  typedef unspecified rk_algorithm_type;
  typedef rk_algorithm_type::coef_a_type coef_a_type;
  typedef rk_algorithm_type::coef_b_type coef_b_type;
  typedef rk_algorithm_type::coef_c_type coef_c_type;

  // construct/copy/destruct
  explicit_generic_rk(const coef_a_type &, const coef_b_type &, const coef_c_type &,
                      const algebra_type &, = algebra_type());

  // public member functions
  template&lt;typename System, typename StateIn, typename DerivIn,
           typename StateOut&gt;
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                      StateOut &, time_type);
  template&lt;typename StateIn&gt; void adjust_size(const StateIn &);
  order_type order(void) const;
  template&lt;typename System, typename StateInOut&gt;
    void do_step(System, StateInOut &, time_type, time_type);
  template&lt;typename System, typename StateInOut&gt;
    void do_step(System, const StateInOut &, time_type, time_type);
  template&lt;typename System, typename StateIn, typename DerivIn&gt;
    boost::disable_if&lt; boost::is_same&lt; DerivIn, time_type &gt;, void &gt;::type
do_step(System, StateIn &, const DerivIn &, time_type, time_type);
  template&lt;typename System, typename StateIn, typename StateOut&gt;
    void do_step(System, const StateIn &, time_type, StateOut &,
                 time_type);
  template&lt;typename System, typename StateIn, typename DerivIn, typename StateOut&gt;
    void do_step(System, const StateIn &, const DerivIn &, time_type,
                 StateOut &, time_type);
  template&lt;typename System, typename StateIn, typename StateOut&gt;
    void do_step(System, const StateIn &, time_type, StateOut &,
                 time_type);
  algebra_type & algebra();
  const algebra_type & algebra() const;

  // private member functions
  template&lt;typename StateIn&gt; bool resize_impl(const StateIn &);
};
```
Description

This class implements the explicit Runge-Kutta algorithms without error estimation in a generic way. The Butcher tableau is passed to the stepper which constructs the stepper scheme with the help of a template-metaprogramming algorithm. ToDo : Add example!

This class derives explicit_stepper_base which provides the stepper interface.

Template Parameters

1. `size_t StageCount`
   The number of stages of the Runge-Kutta algorithm.

2. `size_t Order`
   The order of the stepper.

3. `typename State`
   The type representing the state of the ODE.

4. `typename Value`
   The floating point type which is used in the computations.

5. `typename Deriv`

6. `typename Time`
   The type representing the independent variable - the time - of the ODE.

7. `typename Algebra`
   The algebra type.

8. `typename Operations`
   The operations type.

9. `typename Resizer`
   The resizer policy type.

`explicit_generic_rk` public construct/copy/destruct

1. `explicit_generic_rk(const coef_a_type & a, const coef_b_type & b, const coef_c_type & c, const algebra_type & algebra = algebra_type());`

Constructs the `explicit_generic_rk` class. See examples section for details on the coefficients.

Parameters:  

- `a`  
  Triangular matrix of parameters `b` in the Butcher tableau.
algebra

A copy of algebra is made and stored inside `explicit_stepper_base`.

b

Last row of the butcher tableau.

c

Parameters to calculate the time points in the Butcher tableau.

**explicit_generic_rk public member functions**

1. Template:

   ```cpp
template<typename System, typename StateIn, typename DerivIn,
   typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
   time_type t, StateOut & out, time_type dt);
   ```

   This method performs one step. The derivative `dxdt` of `in` at the time `t` is passed to the method. The result is updated out of place, hence the input is in `in` and the output in `out`. Access to this step functionality is provided by `explicit_stepper_base` and `do_step_impl` should not be called directly.

   Parameters:
   - `dt` The step size.
   - `dxdt` The derivative of `x` at `t`.
   - `in` The state of the ODE which should be solved. `in` is not modified in this method.
   - `out` The result of the step is written in `out`.
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
   - `t` The value of the time, at which the step should be performed.

2. Template:

   ```cpp
template<typename StateIn> void adjust_size(const StateIn & x);
   ```

   Adjust the size of all temporaries in the stepper manually.

   Parameters:
   - `x` A state from which the size of the temporaries to be resized is deduced.

3. Template:

   ```cpp
   order_type order(void) const;
   ```

   Returns: Returns the order of the stepper.

4. Template:

   ```cpp
template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);
   ```

   This method performs one step. It transforms the result in-place.

   Parameters:
   - `dt` The step size.
   - `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
   - `t` The value of the time, at which the step should be performed.
   - `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in `x`.

5. Template:

   ```cpp
template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);
   ```

   Second version to solve the forwarding problem, can be called with Boost.Range as `StateInOut`.

6. Template:

   ```cpp
template<typename System, typename StateInOut, typename DerivIn>
   boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
   do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
   time_type dt);
   ```

   The method performs one step. Additionally to the other method the derivative of `x` is also passed to this method. It is supposed to be used in the following way:
sys( x, dxdt, t );
stepper.do_step( sys, x, dxdt, t, dt );

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.

7. 

```cpp
template<typename System, typename StateIn, typename StateOut>
void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

8. 

```cpp
template<typename System, typename StateIn, typename DerivIn, 
         typename StateOut>
void do_step(System system, const StateIn & in, const DerivIn & dxdt, 
             time_type t, StateOut & out, time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in, dxdt, t );
stepper.do_step( sys, in, dxdt, t, out, dt );
```

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
t

The value of the time, at which the step should be performed.

9. algebra_type & algebra();

Returns: A reference to the algebra which is held by this class.

10. const algebra_type & algebra() const;

Returns: A const reference to the algebra which is held by this class.

**explicit_generic_rk private member functions**

1. template<typename StateIn> bool resize_impl(const StateIn & x);

**Header** `<boost/numeric/odeint/stepper/implicit_euler.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename ValueType, typename Resizer = initially_resizer>
            class implicit_euler;
        }
    }
}
```

**Class template implicit_euler**

`boost::numeric::odeint::implicit_euler`
**Synopsis**

```cpp
// In header: <boost/numeric/odeint/stepper/implicit_euler.hpp>

template<typename ValueType, typename Resizer = initially_resizer>
class implicit_euler {
public:
    // types
    typedef ValueType value_type;
    typedef value_type time_type;
    typedef boost::numeric::ublas::vector< value_type > state_type;
    typedef state_wrapper< state_type > wrapped_state_type;
    typedef state_type deriv_type;
    typedef state_wrapper< deriv_type > wrapped_deriv_type;
    typedef boost::numeric::ublas::matrix< value_type > matrix_type;
    typedef state_wrapper< matrix_type > wrapped_matrix_type;
    typedef boost::numeric::ublas::permutation_matrix< size_t > pmatrix_type;
    typedef state_wrapper< pmatrix_type > wrapped_pmatrix_type;
    typedef Resizer resizer_type;
    typedef stepper_tag stepper_category;
    typedef implicit_euler< ValueType, Resizer > stepper_type;

    // construct/copy/destruct
    implicit_euler(value_type epsilon = 1E-6);

    // public member functions
    template<typename System>
    void do_step(System system, state_type & x, time_type t, time_type dt);
    template<typename StateType> void adjust_size(const StateType & x);

    // private member functions
    template<typename StateIn> bool resize_impl(const StateIn & x);
    void solve(state_type & x, matrix_type & m);
};
```

**Description**

**implicit_euler public construct/copy/destruct**

1. `implicit_euler(value_type epsilon = 1E-6);`

**implicit_euler public member functions**

1. `template<typename System>
void do_step(System system, state_type & x, time_type t, time_type dt);`

2. `template<typename StateType> void adjust_size(const StateType & x);`

**implicit_euler private member functions**

1. `template<typename StateIn> bool resize_impl(const StateIn & x);`

2. `void solve(state_type & x, matrix_type & m);`
Class template modified_midpoint

boost::numeric::odeint::modified_midpoint
Synopsis

// In header: <boost/numeric/odeint/stepper/modified_midpoint.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class modified_midpoint : public boost::numeric::odeint::explicit_stepper_base< Stepper, Order, 
State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
public:

    // types
    typedef explicit_stepper_base< modified_midpoint< State, Value, Deriv, Time, Algebra, Operations, Resizer >, 2, State, Value, Deriv, Time, Algebra, Operations, Resizer > stepper_base_type;
    typedef stepper_base_type::state_type state_type;
    typedef stepper_base_type::wrapped_state_type wrapped_state_type;
    typedef stepper_base_type::value_type value_type;
    typedef stepper_base_type::deriv_type deriv_type;
    typedef stepper_base_type::wrapped_deriv_type wrapped_deriv_type;
    typedef stepper_base_type::time_type time_type;
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::operations_type operations_type;
    typedef stepper_base_type::resizer_type resizer_type;
    typedef stepper_base_type::stepper_type stepper_type;

    // construct/copy/destruct
    modified_midpoint(unsigned short = 2, const algebra_type & = algebra_type());

    // public member functions
    template<typename System, typename StateIn, typename DerivIn, 
             typename StateOut>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type, 
                       StateOut &, time_type);
    void set_steps(unsigned short);
    unsigned short steps(void) const;
    template<typename StateIn> void adjust_size(const StateIn &);
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut>
    void do_step(System, const StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut, typename DerivIn>
    boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type 
    do_step(System, StateInOut &, const DerivIn &, time_type, time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, StateOut &, time_type);
    template<typename System, typename StateIn, typename DerivIn, 
             typename StateOut>
    void do_step(System, const StateIn &, const DerivIn &, time_type, 
                 StateOut &, time_type);

    // XML to PDF by RenderX XEP XSL-FO Formatter, visit us at http://www.renderx.com/
Description

Implementation of the modified midpoint method with a configurable number of intermediate steps. This class is used by the Bulirsch-Stoer algorithm and is not meant for direct usage.

**modified_midpoint** public construct/copy/destruct

1. `modified_midpoint(unsigned short steps = 2,
                       const algebra_type & algebra = algebra_type());`

**modified_midpoint** public member functions

1. `template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                     time_type t, StateOut & out, time_type dt);`

2. `void set_steps(unsigned short steps);`

3. `unsigned short steps(void) const;`

4. `template<typename StateIn>
   void adjust_size(const StateIn & x);`

5. `order_type order(void) const;`

   Returns: Returns the order of the stepper.

6. `template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);`

   This method performs one step. It transforms the result in-place.

   Parameters:
   
   - `dt` The step size.
   - `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
   - `t` The value of the time, at which the step should be performed.
   - `x` The state of the ODE which should be solved. After calling do_step the result is updated in x.

7. `template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);`

   Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.
8.  

```cpp
template<typename System, typename StateInOut, typename DerivIn>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
time_type dt);
```

The method performs one step. Additionally to the other method the derivative of \( x \) is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x, dxdt, t );
stepper.do_step( sys, x, dxdt, t, dt );
```

The result is updated in place in \( x \). This method is disabled if \( \text{Time} \) and \( \text{Deriv} \) are of the same type. In this case the method could not be distinguished from other `do_step` versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `dxdt` The derivative of \( x \) at \( t \).
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in \( x \).

9.  

```cpp
template<typename System, typename StateIn, typename StateOut>
void do_step(System system, const StateIn & in, time_type t, StateOut & out,
time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place.

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `in` The state of the ODE which should be solved. \( \text{in} \) is not modified in this method.
- `out` The result of the step is written in \( \text{out} \).
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.

10.  

```cpp
template<typename System, typename StateIn, typename DerivIn,
        typename StateOut>
void do_step(System system, const StateIn & in, const DerivIn & dxdt,
time_type t, StateOut & out, time_type dt);
```

The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of \( x \) at \( t \) is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in, dxdt, t );
stepper.do_step( sys, in, dxdt, t, out, dt );
```
This method does not solve the forwarding problem.

Parameters:
- \( dt \) The step size.
- \( dxdt \) The derivative of \( x \) at \( t \).
- \( in \) The state of the ODE which should be solved. \( in \) is not modified in this method.
- \( out \) The result of the step is written in \( out \).
- \( system \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \) The value of the time, at which the step should be performed.

Returns:

11. `algebra_type & algebra();`
   
   Returns: A reference to the algebra which is held by this class.

12. `const algebra_type & algebra() const;`
   
   Returns: A const reference to the algebra which is held by this class.

`modified_midpoint` private member functions

1. `template<typename StateIn> bool resize_impl(const StateIn & x);`

Class template `modified_midpoint_dense_out`

`boost::numeric::odeint::modified_midpoint_dense_out`
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/modified_midpoint.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class modified_midpoint_dense_out {
public:
    // types
    typedef State state_type;
    typedef Value value_type;
    typedef Deriv deriv_type;
    typedef Time time_type;
    typedef Algebra algebra_type;
    typedef Operations operations_type;
    typedef Resizer resizer_type;
    typedef state_wrapper< state_type > wrapped_state_type;
    typedef state_wrapper< deriv_type > wrapped_deriv_type;
    typedef modified_midpoint_dense_out< State, Value, Deriv, Time, Algebra, Operations, Resizer > stepper_type;
    typedef std::vector< wrapped_deriv_type > deriv_table_type;

    // construct/copy/destruct
    modified_midpoint_dense_out(unsigned short = 2, const algebra_type & = algebra_type());

    // public member functions
    template<typename System, typename StateIn, typename DerivIn, typename StateOut>
    void do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type, state_type &, deriv_table_type &);
    void set_steps(unsigned short);
    unsigned short steps(void) const;
    template<typename StateIn> bool resize(const StateIn &);
    template<typename StateIn> void adjust_size(const StateIn &);
};
```

Description

Implementation of the modified midpoint method with a configurable number of intermediate steps. This class is used by the dense output Bulirsch-Stoer algorithm and is not meant for direct usage.

Note

This stepper is for internal use only and does not meet any stepper concept.
modified_midpoint_dense_out public construct/copy/destruct

1. modified_midpoint_dense_out(unsigned short steps = 2,
   const algebra_type & algebra = algebra_type());

modified_midpoint_dense_out public member functions

1. template<typename System, typename StateIn, typename DerivIn,
   typename StateOut>
   void do_step(System system, const StateIn & in, const DerivIn & dxdt,
   time_type t, StateOut & out, time_type dt, state_type & x_mp,
   deriv_table_type & derivs);

2. void set_steps(unsigned short steps);

3. unsigned short steps(void) const;

4. template<typename StateIn> bool resize(const StateIn & x);

5. template<typename StateIn> void adjust_size(const StateIn & x);

Header <boost/numeric/odeint/stepper/rosenbrock4.hpp>

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Value> struct default_rosenbrock_coefficients;

            template<typename Value,
                typename Coefficients = default_rosenbrock_coefficients< Value >,
                typename Resizer = initially_resizer
            
            class rosenbrock4;
            }
        }
    }
}

Struct template default_rosenbrock_coefficients

boost::numeric::odeint::default_rosenbrock_coefficients
Synopsis

// In header: <boost/numeric/odeint/stepper/rosenbrock4.hpp>

namespace boost { namespace numeric { namespace odeint { namespace stepper {

   template<typename Value>
   struct default_rosenbrock_coefficients {

      // types
      typedef Value value_type;
      typedef unsigned short order_type;

      // construct/copy/destruct
      default_rosenbrock_coefficients(void);

      // public data members
      const value_type gamma;
      const value_type d1;
      const value_type d2;
      const value_type d3;
      const value_type d4;
      const value_type c2;
      const value_type c3;
      const value_type c4;
      const value_type d11;
      const value_type d12;
      const value_type c21;
      const value_type c22;
      const value_type a21;
      const value_type a22;
      const value_type c31;
      const value_type c32;
      const value_type a31;
      const value_type a32;
      const value_type c41;
      const value_type c42;
      const value_type c43;
      const value_type d21;
      const value_type d22;
      const value_type d23;
      const value_type d24;
      const value_type d31;
      const value_type d32;
      const value_type d33;
      const value_type d34;
      const value_type c51;
      const value_type c52;
      const value_type c53;
      const value_type c54;
      const value_type a51;
      const value_type a52;
      const value_type a53;
      const value_type a54;
      static const order_type stepper_order;
      static const order_type error_order;
   };

}}}}}}

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Description

default_rosenbrock_coefficients public construct/copy/destruct

1. default_rosenbrock_coefficients(void);

Class template rosenbrock4

boost::numeric::odeint::rosenbrock4

Synopsis

```
// In header: <boost/numeric/odeint/stepper/rosenbrock4.hpp>

template<typename Value,
         typename Coefficients = default_rosenbrock_coefficients< Value >,
         typename Resizer = initially_resizer>
class rosenbrock4 {
public:
  // types
  typedef Value value_type;
  typedef boost::numeric::ublas::vector< value_type > state_type;
  typedef value_type time_type;
  typedef boost::numeric::ublas::matrix< value_type > matrix_type;
  typedef boost::numeric::ublas::permutation_matrix< size_t > pmatrix_type;
  typedef Coefficients rosenbrock_coefficients;
  typedef stepper_tag stepper_category;
  typedef unsigned short order_type;
  typedef state_wrapper< state_type > wrapped_state_type;
  typedef state_wrapper< deriv_type > wrapped_deriv_type;
  typedef state_wrapper< matrix_type > wrapped_matrix_type;
  typedef state_wrapper< pmatrix_type > wrapped_pmatrix_type;
  typedef rosenbrock4< Value, Coefficients, Resizer > stepper_type;

  // construct/copy/destruct
  rosenbrock4(void);

  // public member functions
  order_type order() const;
  template<typename System>
  void do_step(System, const state_type &, time_type, state_type &, time_type);
  template<typename System>
  void do_step(System, state_type &, time_type, time_type, state_type &);
  template<typename System>
  void do_step(System, const state_type &, time_type, state_type &);
  template<typename System>
  void do_step(System, state_type &, time_type, time_type);
  void prepare_dense_output();
  void calc_state(time_type, state_type &, const state_type &, time_type, const state_type &, time_type);
  template<typename StateType>
  void adjust_size(const StateType &);

  // protected member functions
  template<typename StateIn>
  bool resize_impl(const StateIn &);
```

```
Description

rosenbrock4 public construct/copy/destruct

1. rosenbrock4(void);

rosenbrock4 public member functions

1. order_type order() const;

2. template<typename System>
   void do_step(System system, const state_type & x, time_type t,
                state_type & xout, time_type dt, state_type & xerr);

3. template<typename System>
   void do_step(System system, state_type & x, time_type t, time_type dt,
                state_type & xerr);

4. template<typename System>
   void do_step(System system, const state_type & x, time_type t,
                state_type & xout, time_type dt);

5. template<typename System>
   void do_step(System system, state_type & x, time_type t, time_type dt);

6. void prepare_dense_output();

7. void calc_state(time_type t, state_type & x, const state_type & x_old,
                   time_type t_old, const state_type & x_new, time_type t_new);

8. template<typename StateType> void adjust_size(const StateType & x);

rosenbrock4 protected member functions

1. template<typename StateIn> bool resize_impl(const StateIn & x);
2. `template<typename StateIn> bool resize_x_err(const StateIn & x);`

Header `<boost/numeric/odeint/stepper/rosenbrock4_controller.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Stepper> class rosenbrock4_controller;
        }
    }
}
```

**Class template rosenbrock4_controller**

boost::numeric::odeint::rosenbrock4_controller
Synopsis

// In header: <boost/numeric/odeint/stepper/rosenbrock4_controller.hpp>

```cpp
template<typename Stepper>
class rosenbrock4_controller {
public:
    // types
    typedef Stepper stepper_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::wrapped_state_type wrapped_state_type;
    typedef stepper_type::time_type time_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::wrapped_deriv_type wrapped_deriv_type;
    typedef stepper_type::resizer_type resizer_type;
    typedef controlled_stepper_tag stepper_category;
    typedef rosenbrock4_controller< Stepper > controller_type;

    // construct/copy/destruct
    rosenbrock4_controller(value_type atol = 1.0e-6, value_type rtol = 1.0e-6, const stepper_type & stepper = stepper_type());

    // public member functions
    value_type error(const state_type &, const state_type &, const state_type &);
    value_type last_error(void) const;
    template<typename System>
    boost::numeric::odeint::controlled_step_result try_step(System, state_type &, time_type &, time_type &);
    template<typename System>
    boost::numeric::odeint::controlled_step_result try_step(System, const state_type &, time_type &, state_type &, time_type &);
    template<typename StateType> void adjust_size(const StateType &);
    stepper_type & stepper(void);
    const stepper_type & stepper(void) const;

    // private member functions
    template<typename StateIn> bool resize_m_xerr(const StateIn &);
    template<typename StateIn> bool resize_m_xnew(const StateIn &);
};
```

Description

**rosenbrock4_controller public construct/copy/destruct**

1. rosenbrock4_controller(value_type atol = 1.0e-6, value_type rtol = 1.0e-6, const stepper_type & stepper = stepper_type());

**rosenbrock4_controller public member functions**

1. value_type error(const state_type & x, const state_type & xold, const state_type & xerr);

2. value_type last_error(void) const;
3. template<typename System>
   boost::numeric::odeint::controlled_step_result
   try_step(System sys, state_type & x, time_type & t, time_type & dt);

4. template<typename System>
   boost::numeric::odeint::controlled_step_result
   try_step(System sys, const state_type & x, time_type & t, state_type & xout,
            time_type & dt);

5. template<typename StateType> void adjust_size(const StateType & x);

6. stepper_type & stepper(void);

7. const stepper_type & stepper(void) const;

rosenbrock4_controller private member functions

1. template<typename StateIn> bool resize_m_xerr(const StateIn & x);

2. template<typename StateIn> bool resize_m_xnew(const StateIn & x);

Header <boost/numeric/odeint/stepper/rosenbrock4_dense_output.hpp>

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename ControlledStepper> class rosenbrock4_dense_output;
        }
    }
}

Class template rosenbrock4_dense_output

boost::numeric::odeint::rosenbrock4_dense_output
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/rosenbrock4_dense_output.hpp>

template<typename ControlledStepper>
class rosenbrock4_dense_output {
public:
    // types
    typedef ControlledStepper controlled_stepper_type;
    typedef controlled_stepper_type::stepper_type stepper_type;
    typedef stepper_type::value_type value_type;
    typedef stepper_type::state_type state_type;
    typedef stepper_type::wrapped_state_type wrapped_state_type;
    typedef stepper_type::time_type time_type;
    typedef stepper_type::deriv_type deriv_type;
    typedef stepper_type::wrapped_deriv_type wrapped_deriv_type;
    typedef stepper_type::resizer_type resizer_type;
    typedef dense_output_stepper_tag stepper_category;
    typedef rosenbrock4_dense_output<ControlledStepper> dense_output_stepper_type;

    // construct/copy/destruct
    rosenbrock4_dense_output(const controlled_stepper_type & = controlled_stepper_type);

    // public member functions
    template<typename StateOut> void calc_state(time_type, StateOut &);
    template<typename StateOut> void calc_state(time_type, const StateOut &);
    template<typename StateType> void adjust_size(const StateType &);
    const state_type & current_state(void) const;
    time_type current_time(void) const;
    const state_type & previous_state(void) const;
    time_type previous_time(void) const;
    time_type current_time_step(void) const;

    // private member functions
    state_type & get_current_state(void);
    const state_type & get_current_state(void) const;
    state_type & get_old_state(void);
    const state_type & get_old_state(void) const;
    void toggle_current_state(void);
    template<typename StateIn> bool resize_impl(const StateIn &);
};
```

Description

**rosenbrock4_dense_output**  
**public construct/copy/destruct**

1. rosenbrock4_dense_output(const controlled_stepper_type & stepper = controlled_stepper_type());

**rosenbrock4_dense_output**  
**public member functions**

1. template<typename StateType>
   void initialize(const StateType & x0, time_type t0, time_type dt0);
template<typename System>
std::pair< time_type, time_type > do_step(System system);

template<typename StateOut> void calc_state(time_type t, StateOut & x);

template<typename StateOut> void calc_state(time_type t, const StateOut & x);

template<typename StateType> void adjust_size(const StateType & x);

const state_type & current_state(void) const;

time_type current_time(void) const;

const state_type & previous_state(void) const;

time_type previous_time(void) const;

time_type current_time_step(void) const;

rosenbrock4_dense_output private member functions

state_type & get_current_state(void);

const state_type & get_current_state(void) const;

state_type & get_old_state(void);

const state_type & get_old_state(void) const;

void toggle_current_state(void);

template<typename StateIn> bool resize_impl(const StateIn & x);
Header `<boost/numeric/odeint/stepper/runge_kutta4.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                     typename Deriv = State, typename Time = Value,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
            class runge_kutta4;
        }
    }
}
```

**Class template runge_kutta4**

boost::numeric::odeint::runge_kutta4 — The classical Runge-Kutta stepper of fourth order.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/runge_kutta4.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class runge_kutta4 : public boost::numeric::odeint::explicit_generic_rk< StageCount, Order, J
                  State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
public:
    // types
    typedef stepper_base_type::state_type state_type;
    typedef stepper_base_type::value_type value_type;
    typedef stepper_base_type::deriv_type deriv_type;
    typedef stepper_base_type::time_type time_type;
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::operations_type operations_type;
    typedef stepper_base_type::resizer_type resizer_type;

    // construct/copy/destruct
    runge_kutta4(const algebra_type & = algebra_type());

    // public member functions
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                       StateOut &, time_type);
    template<typename StateIn> void adjust_size(const StateIn &);
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, StateOut &, time_type);
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut> boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System, const StateIn &, const DerivIn &, time_type,
             StateOut &, time_type);
    template<typename System, typename StateIn, typename StateOut, typename DerivIn>
    void do_step(System, const StateIn &, const DerivIn &, time_type,
                 StateOut &, time_type, algebra_type & algebra());
    const algebra_type & algebra();
};
```

Description

The Runge-Kutta method of fourth order is one standard method for solving ordinary differential equations and is widely used, see also [en.wikipedia.org/wiki/Runge-Kutta_methods](http://en.wikipedia.org/wiki/Runge-Kutta_methods). The method is explicit and fulfills the Stepper concept. Step size control or continuous output are not provided.

This class derives from `explicit_stepper_base` and inherits its interface via CRTP (current recurring template pattern). Furthermore, it derives from `explicit_generic_rk` which is a generic Runge-Kutta algorithm. For more details see `explicit_stepper_base` and `explicit_generic_rk`.
Template Parameters

1. \texttt{typename State}
   The state type.

2. \texttt{typename Value = double}
   The value type.

3. \texttt{typename Deriv = State}
   The type representing the time derivative of the state.

4. \texttt{typename Time = Value}
   The time representing the independent variable - the time.

5. \texttt{typename Algebra = range_algebra}
   The algebra type.

6. \texttt{typename Operations = default_operations}
   The operations type.

7. \texttt{typename Resizer = initially_resizer}
   The resizer policy type.

\texttt{runge_kutta4} public construct/copy/destruct

1. \texttt{runge_kutta4(const algebra_type \& algebra = algebra_type());}
   Constructs the \texttt{runge_kutta4} class. This constructor can be used as a default constructor if the algebra has a default constructor.
   Parameters: algebra A copy of algebra is made and stored inside \texttt{explicit_stepper_base}.

\texttt{runge_kutta4} public member functions

1. \begin{verbatim}
   template<typename System, typename StateIn, typename DerivIn, 
   typename StateOut>
   void do_step_impl(System system, const StateIn \& in, const DerivIn \& dxdt, 
   time_type t, StateOut \& out, time_type dt);
\end{verbatim}
   This method performs one step. The derivative $dxdt$ of $in$ at the time $t$ is passed to the method. The result is updated out of place, hence the input is in $in$ and the output in $out$. Access to this step functionality is provided by \texttt{explicit_stepper_base} and \texttt{do_step_impl} should not be called directly.
   Parameters: dt The step size.
   dxdt The derivative of $x$ at $t$.
   in The state of the ODE which should be solved. in is not modified in this method
   out The result of the step is written in $out$. 

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The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.

The value of the time, at which the step should be performed.

Adjust the size of all temporaries in the stepper manually.

Parameters:
- x A state from which the size of the temporaries to be resized is deduced.

Returns: Returns the order of the stepper.

This method performs one step. It transforms the result in-place.

Parameters:
- dt The step size.
- system The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- t The value of the time, at which the step should be performed.
- x The state of the ODE which should be solved. After calling do_step the result is updated in x.

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

The method performs one step. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

Note

This method does not solve the forwarding problem.

Parameters:
- dt The step size.
- dxdt The derivative of x at t.
- system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- t The value of the time, at which the step should be performed.
- x The state of the ODE which should be solved. After calling do_step the result is updated in x.
7. template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out,
               time_type dt);

   The method performs one step. The state of the ODE is updated out-of-place.

   **Note**
   This method does not solve the forwarding problem.

   Parameters:
   - **dt** The step size.
   - **in** The state of the ODE which should be solved. in is not modified in this method.
   - **out** The result of the step is written in out.
   - **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
   - **t** The value of the time, at which the step should be performed.

8. template<typename System, typename StateIn, typename DerivIn,
          typename StateOut>
   void do_step(System system, const StateIn & in, const DerivIn & dxdt,
                time_type t, StateOut & out, time_type dt);

   The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

   ```cpp
   sys( in , dxdt , t );
   stepper.do_step( sys , in , dxdt , t , out , dt );
   ```

   **Note**
   This method does not solve the forwarding problem.

   Parameters:
   - **dt** The step size.
   - **dxdt** The derivative of x at t.
   - **in** The state of the ODE which should be solved. in is not modified in this method.
   - **out** The result of the step is written in out.
   - **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
   - **t** The value of the time, at which the step should be performed.

9. algebra_type & algebra();

   Returns: A reference to the algebra which is held by this class.

10. const algebra_type & algebra() const;

    Returns: A const reference to the algebra which is held by this class.
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                     typename Deriv = State, typename Time = Value,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
            class runge_kutta4_classic;
        }
    }
}
Synopsis

// In header: <boost/numeric/odeint/stepper/runge_kutta4_classic.hpp>

template<typename State, typename Value = double, typename Deriv = State, 
typename Time = Value, typename Algebra = range_algebra, 
typename Operations = default_operations, 
typename Resizer = initially_resizer>
class runge_kutta4_classic : public boost::numeric::odeint::explicit_stepper_base< Stepper, Order, State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
public:
  // types
  typedef explicit_stepper_base< runge_kutta4_classic< ... >, ... > stepper_base_type;
  typedef stepper_base_type::state_type state_type;
  typedef stepper_base_type::value_type value_type;
  typedef stepper_base_type::deriv_type deriv_type;
  typedef stepper_base_type::time_type time_type;
  typedef stepper_base_type::algebra_type algebra_type;
  typedef stepper_base_type::operations_type operations_type;
  typedef stepper_base_type::resizer_type resizer_type;

  // construct/copy/destroy
  runge_kutta4_classic(const algebra_type & = algebra_type());

  // public member functions
  template<typename System, typename StateIn, typename DerivIn, 
           typename StateOut>
  void do_step_impl(System, const StateIn &, const DerivIn &, time_type, 
                     StateOut &, time_type);
  template<typename StateType> void adjust_size(const StateType &);
  order_type order(void) const;
  template<typename System, typename StateInOut>
  void do_step(System, StateInOut &, time_type, time_type);
  template<typename System, typename StateIn, typename StateOut>
  void do_step(System, const StateIn &, time_type, StateOut &, time_type);
  template<typename System, typename StateIn, typename DerivIn, 
           typename StateOut>
  void do_step(System, const StateIn &, const DerivIn &, time_type, 
               StateOut &, time_type);
  template<typename StateIn> void adjust_size(const StateIn &);
  algebra_type & algebra();
  const algebra_type & algebra() const;

  // private member functions
  template<typename StateIn> bool resize_impl(const StateIn &);
};

Description

The Runge-Kutta method of fourth order is one standard method for solving ordinary differential equations and is widely used, see also en.wikipedia.org/wiki/Runge-Kutta_methods The method is explicit and fulfills the Stepper concept. Step size control or continuous output are not provided. This class implements the method directly, hence the generic Runge-Kutta algorithm is not used.

This class derives from explicit_stepper_base and inherits its interface via CRTP (current recurring template pattern). For more details see explicit_stepper_base.
Template Parameters

1. **typename State**
   The state type.

2. **typename Value = double**
   The value type.

3. **typename Deriv = State**
   The type representing the time derivative of the state.

4. **typename Time = Value**
   The time representing the independent variable - the time.

5. **typename Algebra = range_algebra**
   The algebra type.

6. **typename Operations = default_operations**
   The operations type.

7. **typename Resizer = initially_resizer**
   The resizer policy type.

**runge_kutta4_classic public construct/copy/destroy**

1. ```cpp
   runge_kutta4_classic(const algebra_type & algebra = algebra_type());
```

   Constructs the `runge_kutta4_classic` class. This constructor can be used as a default constructor if the algebra has a default constructor.

   Parameters:
   - `algebra` A copy of algebra is made and stored inside `explicit_stepper_base`.

**runge_kutta4_classic public member functions**

1. ```cpp
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                     time_type t, StateOut & out, time_type dt);
```

   This method performs one step. The derivative $dx/dt$ of $in$ at the time $t$ is passed to the method. The result is updated out of place, hence the input is in $in$ and the output in $out$. Access to this step functionality is provided by `explicit_stepper_base` and `do_step_impl` should not be called directly.

   Parameters:
   - `dt` The step size.
   - `dxdt` The derivative of $x$ at $t$.
   - `in` The state of the ODE which should be solved. $in$ is not modified in this method.
The result of the step is written in out.

The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.

The value of the time, at which the step should be performed.

```cpp
template<typename StateType> void adjust_size(const StateType & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters:
- **x**: A state from which the size of the temporaries to be resized is deduced.

```cpp
order_type order(void) const;
```

Returns: Returns the order of the stepper.

```cpp
template<typename System, typename StateInOut>
void do_step(System system, StateInOut & x, time_type t, time_type dt);
```

This method performs one step. It transforms the result in-place.

Parameters:
- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.

```cpp
template<typename System, typename StateInOut>
void do_step(System system, const StateInOut & x, time_type t, time_type dt);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

```cpp
template<typename System, typename StateInOut, typename DerivIn>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt);
```

The method performs one step. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.
7. template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out,
   time_type dt);

The method performs one step. The state of the ODE is updated out-of-place.

Note
This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **in** The state of the ODE which should be solved. In is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.

8. template<typename System, typename StateIn, typename DerivIn, 
   typename StateOut>
   void do_step(System system, const StateIn & in, const DerivIn & dxdt,
   time_type t, StateOut & out, time_type dt);

The method performs one step. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```
sys( in, dxdt, t );
stepper.do_step( sys, in, dxdt, t, out, dt );
```

Note
This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **in** The state of the ODE which should be solved. In is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.

9. template<typename StateIn> void adjust_size(const StateIn & x);

Adjust the size of all temporaries in the stepper manually.

Parameters:
- **x** A state from which the size of the temporaries to be resized is deduced.

10. algebra_type & algebra();

Returns: A reference to the algebra which is held by this class.

11. const algebra_type & algebra() const;

Returns: A const reference to the algebra which is held by this class.
runge_kutta4_classic private member functions

1. template<typename StateIn> bool resize_impl(const StateIn & x);

Header `<boost/numeric/odeint/stepper/runge_kutta_cash_karp54.hpp>`

namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double, 
                typename Deriv = State, typename Time = Value, 
                typename Algebra = range_algebra, 
                typename Operations = default_operations, 
                typename Resizer = initially_resizer>
                class runge_kutta_cash_karp54;
        }
    }
}

Class template runge_kutta_cash_karp54

boost::numeric::odeint::runge_kutta_cash_karp54 — The Runge-Kutta Cash-Karp method.
Synopsis

// In header: <boost/numeric/odeint/stepper/runge_kutta_cash_karp54.hpp>

template<typename State, typename Value = double, typename Deriv = State,
        typename Time = Value, typename Algebra = range_algebra,
        typename Operations = default_operations,
        typename Resizer = initially_resizer>
class runge_kutta_cash_karp54 : public boost::numeric::odeint::explicit_error_generic_rk< Stage,
        Count, Order, StepperOrder, ErrorOrder, State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
    public:
        // types
        typedef stepper_base_type::state_type state_type;
        typedef stepper_base_type::value_type value_type;
        typedef stepper_base_type::deriv_type deriv_type;
        typedef stepper_base_type::time_type time_type;
        typedef stepper_base_type::algebra_type algebra_type;
        typedef stepper_base_type::operations_type operations_type;
        typedef stepper_base_type::resizer_type resizer_type;

        // construct/copy/destruct
        runge_kutta_cash_karp54(const algebra_type & = algebra_type());

        // public member functions
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut>
        void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                           StateOut &, time_type);
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut>
        void do_step(System, const StateIn &, const DerivIn &, time_type,
                     StateOut &, time_type);
        template<typename System, typename StateIn>
        void adjust_size(const StateIn &);
        order_type order(void) const;
        order_type stepper_order(void) const;
        order_type error_order(void) const;
        template<typename System, typename StateIn, typename StateOut>
        void do_step(System, StateIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename StateOut>
        void do_step(System, const StateIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename StateOut, typename DerivIn>
        boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename StateOut, typename DerivIn>
        boost::disable_if< boost::is_same< StateIn, time_type >, void >::type
        do_step(System, const StateIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename StateOut, typename DerivIn,
                 typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type,
                 Error &);
        template<typename System, typename StateIn, typename StateOut, typename Error>
        do_step(System, StateIn &, const DerivIn &, time_type, StateOut &, time_type,
void do_step(System, const StateIn &, time_type, StateOut &, time_type, Err &);

template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err>
void do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type, Err &);

algebra_type & algebra();
const algebra_type & algebra() const;

Description

The Runge-Kutta Cash-Karp method is one of the standard methods for solving ordinary differential equations, see en.wikipedia.org/wiki/Cash-Karp_methods. The method is explicit and fulfills the Error Stepper concept. Step size control is provided but continuous output is not available for this method.

This class derives from explicit_error_stepper_base and inherits its interface via CRTP (current recurring template pattern). Furthermore, it derivs from explicit_error_generic_rk which is a generic Runge-Kutta algorithm with error estimation. For more details see explicit_error_stepper_base and explicit_error_generic_rk.

Template Parameters

1. typename State

The state type.

2. typename Value = double

The value type.

3. typename Deriv = State

The type representing the time derivative of the state.

4. typename Time = Value

The time representing the independent variable - the time.

5. typename Algebra = range_algebra

The algebra type.

6. typename Operations = default_operations

The operations type.

7. typename Resizer = initially_resizer

The resizer policy type.

runge_kutta_cash_karp54 public construct/copy/destroy

1. runge_kutta_cash_karp54(const algebra_type & algebra = algebra_type());
Constructs the `runge_kutta_cash_karp54` class. This constructor can be used as a default constructor if the algebra has a default constructor.

**Parameters:**
- `algebra` A copy of algebra is made and stored inside `explicit_stepper_base`.

**runge_kutta_cash_karp54 public member functions**

1. ```
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut, typename Err>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                     time_type t, StateOut & out, time_type dt, Err & xerr);
```  
   This method performs one step. The derivative `dxdt` of `in` at the time `t` is passed to the method. The result is updated out-of-place, hence the input is in `in` and the output in `out`. Furthermore, an estimation of the error is stored in `xerr`. `do_step_impl` is used by `explicit_error_stepper_base`.

   **Parameters:**
   - `dt` The step size.
   - `dxdt` The derivative of `x` at `t`.
   - `in` The state of the ODE which should be solved. `in` is not modified in this method.
   - `out` The result of the step is written in `out`.
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
   - `t` The value of the time, at which the step should be performed.
   - `xerr` The result of the error estimation is written in `xerr`.

2. ```
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                     time_type t, StateOut & out, time_type dt);
```  
   This method performs one step. The derivative `dxdt` of `in` at the time `t` is passed to the method. The result is updated out-of-place, hence the input is in `in` and the output in `out`. Access to this step functionality is provided by `explicit_stepper_base` and `do_step_impl` should not be called directly.

   **Parameters:**
   - `dt` The step size.
   - `dxdt` The derivative of `x` at `t`.
   - `in` The state of the ODE which should be solved. `in` is not modified in this method.
   - `out` The result of the step is written in `out`.
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
   - `t` The value of the time, at which the step should be performed.

3. ```
   template<typename StateIn> void adjust_size(const StateIn & x);
```  
   Adjust the size of all temporaries in the stepper manually.

   **Parameters:**
   - `x` A state from which the size of the temporaries to be resized is deduced.

4. ```
   order_type order(void) const;
```  
   **Returns:** Returns the order of the stepper if it used without error estimation.

5. ```
   order_type stepper_order(void) const;
```  
   **Returns:** Returns the order of a step if the stepper is used without error estimation.

6. ```
   order_type error_order(void) const;
```
Returns: Returns the order of an error step if the stepper is used without error estimation.

7. 

```cpp
template<typename System, typename StateInOut>
void do_step(System system, StateInOut & x, time_type t, time_type dt);
```

This method performs one step. It transforms the result in-place.

Parameters:
- `dt` The step size.
- `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling do_step the result is updated in x.

8. 

```cpp
template<typename System, typename StateInOut>
void do_step(System system, const StateInOut & x, time_type t, time_type dt);
```

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

9. 

```cpp
template<typename System, typename StateInOut, typename DerivIn>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
         time_type dt);
```

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `dxdt` The derivative of x at t.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling do_step the result is updated in x.

10. 

```cpp
template<typename System, typename StateIn, typename StateOut>
boost::disable_if< boost::is_same< StateIn, time_type >, void >::type
do_step(System system, const StateIn & in, time_type t, StateOut & out,
         time_type dt);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other do_step variants.

**Note**

This method does not solve the forwarding problem.
Parameters:  
- \( dt \): The step size.  
- \( in \): The state of the ODE which should be solved. \( in \) is not modified in this method.  
- \( out \): The result of the step is written in \( out \).  
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
- \( t \): The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of \( x \) at \( t \) is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if \( \text{DerivIn} \) and \( \text{Time} \) are of same type.

**Note**

This method does not solve the forwarding problem.

Parameters:  
- \( dt \): The step size.  
- \( dxdt \): The derivative of \( x \) at \( t \).  
- \( in \): The state of the ODE which should be solved. \( in \) is not modified in this method.  
- \( out \): The result of the step is written in \( out \).  
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
- \( t \): The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

Parameters:  
- \( dt \): The step size.  
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
- \( t \): The value of the time, at which the step should be performed.  
- \( x \): The state of the ODE which should be solved. \( x \) is updated by this method.  
- \( xerr \): The estimation of the error is stored in \( xerr \).

Second version to solve the forwarding problem, can be called with Boost.Range as \( \text{StateInOut} \).

Parameters:  
- \( dt \): The step size.  
- \( system \): The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
- \( t \): The value of the time, at which the step should be performed.  
- \( dxdt \): The derivative of \( x \) at \( t \).  
- \( xerr \): The estimation of the error is stored in \( xerr \).
The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt , xerr );
```

The result is updated in place in x. This method is disabled if Time and DerivIn are of the same type. In this case the method could not be distinguished from other `do_step` versions.

**Note**

This method does not solve the forwarding problem.

Parameters:

- `dt` The step size.
- `dxdt` The derivative of x at t.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in x.
- `xerr` The error estimate is stored in x.

```cpp
template<typename System, typename StateIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

**Note**

This method does not solve the forwarding problem.

Parameters:

- `dt` The step size.
- `in` The state of the ODE which should be solved. `in` is not modified in this method.
- `out` The result of the step is written in `out`.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `xerr` The error estimate.

```cpp
template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, const DerivIn & dxdt, time_type t, StateOut & out, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if DerivIn and Time are of same type.
This method does not solve the forwarding problem.

Parameters:

- `dt`: The step size.
- `dxdt`: The derivative of x at t.
- `in`: The state of the ODE which should be solved. in is not modified in this method
- `out`: The result of the step is written in out.
- `system`: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t`: The value of the time, at which the step should be performed.
- `xerr`: The error estimate.

17. `algebra_type & algebra();`

Returns: A reference to the algebra which is held by this class.

18. `const algebra_type & algebra() const;`

Returns: A const reference to the algebra which is held by this class.

**Header** `<boost/numeric/odeint/step-per/runge_kutta_cash_karp54_classic.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                     typename Deriv = State, typename Time = double,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
                class runge_kutta_cash_karp54_classic;
        }
    }
}
```

**Class template runge_kutta_cash_karp54_classic**

Boost::numeric::odeint::runge_kutta_cash_karp54_classic — The Runge-Kutta Cash-Karp method implemented without the generic Runge-Kutta algorithm.
Synopsis

// In header: <boost/numeric/odeint/stepper/runge_kutta_cash_karp54_classic.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = double, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class runge_kutta_cash_karp54_classic : public boost::numeric::odeint::explicit_error_steppreter_base< Stepper, Order, StepperOrder, ErrorOrder, State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
    public:
        // types
        typedef explicit_error_stepper_base< runge_kutta_cash_karp54_classic< ... >, ... > step_per_base_type;
        typedef stepper_base_type::state_type state_type;
        typedef stepper_base_type::value_type value_type;
        typedef stepper_base_type::deriv_type deriv_type;
        typedef stepper_base_type::time_type time_type;
        typedef stepper_base_type::algebra_type algebra_type;
        typedef stepper_base_type::operations_type operations_type;
        typedef stepper_base_type::resizer_type resizer_type;

        // construct/copy/destuct
        runge_kutta_cash_karp54_classic(const algebra_type & = algebra_type());

        // public member functions
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut, typename Err>
        void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                           StateOut &, time_type, Err &);
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut>
        void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                           StateOut &, time_type);
        template<typename StateIn> void adjust_size(const StateIn &);
        order_type order(void) const;
        order_type stepper_order(void) const;
        order_type error_order(void) const;
        template<typename System, typename StateInOut>
        void do_step(System, StateInOut &, time_type, time_type);
        template<typename System, typename StateInOut>
        void do_step(System, const StateInOut &, time_type, time_type);
        template<typename System, typename StateIn, typename DerivIn>
        boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System, StateIn &, const DerivIn &, time_type, time_type);
        template<typename System, typename StateIn, typename StateOut>
        boost::disable_if< boost::is_same< StateIn, time_type >, void >::type
do_step(System, const StateIn &, time_type, StateOut &, time_type);
        template<typename System, typename StateIn, typename DerivIn,
                 typename StateOut>
        boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &,
                 time_type);
The Runge-Kutta Cash-Karp method is one of the standard methods for solving ordinary differential equations, see [en.wikipedia.org/wiki/Cash-Karp_method](http://en.wikipedia.org/wiki/Cash-Karp_method). The method is explicit and fulfills the Error Stepper concept. Step size control is provided but continuous output is not available for this method.

This class derives from `explicit_error_stepper_base` and inherits its interface via CRTP (current recurring template pattern). This class implements the method directly, hence the generic Runge-Kutta algorithm is not used.

**Template Parameters**

1. `typename State`  
   The state type.

2. `typename Value = double`  
   The value type.

3. `typename Deriv = State`  
   The type representing the time derivative of the state.

4. `typename Time = double`  
   The time representing the independent variable - the time.

5. `typename Algebra = range_algebra`  
   The algebra type.

6. `typename Operations = default_operations`  
   The operations type.
The operations type.

typedef Resizer = initially_resizer

The resizer policy type.

runge_kutta_cash_karp54_classic public construct/copy/destroy

1. runge_kutta_cash_karp54_classic(const algebra_type & algebra = algebra_type());

Constructs the runge_kutta_cash_karp54_classic class. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters: algebra A copy of algebra is made and stored inside explicit_stepper_base.

runge_kutta_cash_karp54_classic public member functions

1. template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt, time_type t, StateOut & out, time_type dt, Err & xerr);

This method performs one step. The derivative $dxdt$ of $in$ at the time $t$ is passed to the method. The result is updated out-of-place, hence the input is in $in$ and the output in $out$. Furthermore, an estimation of the error is stored in $xerr$. Access to this step functionality is provided by explicit_error_stepper_base and do_step_impl should not be called directly.

Parameters: dt The step size.
 dxdt The derivative of $x$ at $t$.
 in The state of the ODE which should be solved. $in$ is not modified in this method.
 out The result of the step is written in $out$.
 system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
 t The value of the time, at which the step should be performed.
 xerr The result of the error estimation is written in $xerr$.

2. template<typename System, typename StateIn, typename DerivIn, typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt, time_type t, StateOut & out, time_type dt);

This method performs one step. The derivative $dxdt$ of $in$ at the time $t$ is passed to the method. The result is updated out-of-place, hence the input is in $in$ and the output in $out$. Access to this step functionality is provided by explicit_error_stepper_base and do_step_impl should not be called directly.

Parameters: dt The step size.
 dxdt The derivative of $x$ at $t$.
 in The state of the ODE which should be solved. $in$ is not modified in this method.
 out The result of the step is written in $out$.
 system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
 t The value of the time, at which the step should be performed.

3. template<typename StateIn> void adjust_size(const StateIn & x);

Adjust the size of all temporaries in the stepper manually.

Parameters: x A state from which the size of the temporaries to be resized is deduced.
4. `order_type order(void) const;`  
Returns: Returns the order of the stepper if it used without error estimation.

5. `order_type stepper_order(void) const;`  
Returns: Returns the order of a step if the stepper is used without error estimation.

6. `order_type error_order(void) const;`  
Returns: Returns the order of an error step if the stepper is used without error estimation.

7. `template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);`  
This method performs one step. It transforms the result in-place.

Parameters:
- `dt` The step size.
- `system` The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling do_step the result is updated in x.

8. `template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);`  
Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

9. `template<typename System, typename StateInOut, typename DerivIn>
   boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
   do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt);`  
The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**
This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `dxdt` The derivative of x at t.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling do_step the result is updated in x.
The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other `do_step` variants.

**Note**

This method does not solve the forwarding problem.

### Parameters:
- **dt** The step size.
- **in** The state of the ODE which should be solved. `in` is not modified in this method.
- **out** The result of the step is written in `out`.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if DerivIn and Time are of same type.

**Note**

This method does not solve the forwarding problem.

### Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **in** The state of the ODE which should be solved. `in` is not modified in this method.
- **out** The result of the step is written in `out`.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

### Parameters:
- **dt** The step size.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **x** The state of the ODE which should be solved. `x` is updated by this method.
The estimation of the error is stored in `xerr`.

```cpp
template<typename System, typename StateInOut, typename Err>
void do_step(System system, const StateInOut & x, time_type t, time_type dt,
             Err & xerr);
```

Second version to solve the forwarding problem, can be called with `Boost.Range` as `StateInOut`.

```cpp
template<typename System, typename StateInOut, typename DerivIn, typename Err>
boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t,
time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by `Stepper`. Additionally to the other method the derivative of `x` is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt , xerr );
```

The result is updated in place in `x`. This method is disabled if `Time` and `DerivIn` are of the same type. In this case the method could not be distinguished from other `do_step` versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `dxdt` The derivative of `x` at `t`.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `x` The state of the ODE which should be solved. After calling `do_step` the result is updated in `x`.
- `xerr` The error estimate is stored in `xerr`.

The method performs one step with the stepper passed by `Stepper`. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

**Note**

This method does not solve the forwarding problem.

Parameters:
- `dt` The step size.
- `in` The state of the ODE which should be solved. `in` is not modified in this method.
- `out` The result of the step is written in `out`.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `xerr` The error estimate.
The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if DerivIn and Time are of same type.

### Note
This method does not solve the forwarding problem.

**Parameters:**
- `dt` The step size.
- `dxdt` The derivative of x at t.
- `in` The state of the ODE which should be solved. in is not modified in this method.
- `out` The result of the step is written in out.
- `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- `t` The value of the time, at which the step should be performed.
- `xerr` The error estimate.

**Returns:**
- `algebra_type & algebra();` A reference to the algebra which is held by this class.
- `const algebra_type & algebra() const;` A const reference to the algebra which is held by this class.

### runge_kutta_cash_karp54_classic private member functions

1. `template<typename StateIn> bool resize_impl(const StateIn & x);`

### Header

```
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double, 
                typename Deriv = State, typename Time = Value, 
                typename Algebra = range_algebra, 
                typename Operations = default_operations, 
                typename Resizer = initially_resizer>
                class runge_kutta_dopri5;
        }
    }
}
```
Class template runge_kutta_dopri5

boost::numeric::odeint::runge_kutta_dopri5 — The Runge-Kutta Dormand-Prince 5 method.

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/runge_kutta_dopri5.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class runge_kutta_dopri5 : public boost::numeric::odeint::explicit_error_stepper_fsal_base< StepPer, Order, StepperOrder, ErrorOrder, State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
public:

    // types
    typedef explicit_error_stepper_fsal_base< runge_kutta_dopri5< ... >, ... > stepper_base_type;
    typedef stepper_base_type::state_type state_type;
    typedef stepper_base_type::value_type value_type;
    typedef stepper_base_type::deriv_type deriv_type;
    typedef stepper_base_type::time_type time_type;
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::operations_type operations_type;
    typedef stepper_base_type::resizer_type resizer_type;

    // construct/copy/destroy
    runge_kutta_dopri5(const algebra_type & = algebra_type());

    // public member functions
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut, typename DerivOut>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                      StateOut &, DerivOut &, time_type);
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut, typename DerivOut, typename Err>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                      StateOut &, DerivOut &, time_type, Err &);
    template<typename StateOut, typename StateIn1, typename DerivIn1,
             typename StateIn2, typename DerivIn2>
    void calc_state(time_type, StateOut &, const StateIn1 &, const DerivIn1 &,
                    time_type, const StateIn2 &, const DerivIn2 &, time_type) const;
    template<typename StateIn> void adjust_size(const StateIn &);
    order_type order(void) const;
    order_type stepper_order(void) const;
    order_type error_order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut>
    void do_step(System, const StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut, typename DerivInOut>
    boost::disable_if< boost::is_same< StateInOut, time_type >, void >::type
do_step(System, StateInOut &, DerivInOut &, time_type, time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, StateOut &, time_type);
    template<typename System, typename StateIn, typename DerivIn, 
             typename StateOut, typename DerivOut>
    void do_step(System, const StateIn &, const DerivIn &, time_type,
                 StateOut &, DerivOut &, time_type);
    template<typename System, typename StateIn, typename StateOut, typename DerivOut>
    void do_step(System, const StateIn &, const DerivIn &, time_type, 
                 StateOut &, DerivOut &, time_type);
    template<typename System, typename StateIn, typename StateOut, typename DerivOut, 
              typename Err>
    void do_step(System, const StateIn &, const DerivIn &, time_type, 
                 StateOut &, DerivOut &, time_type, Err &);

```

Description

The Runge-Kutta Dormand-Prince 5 method is a very popular method for solving ODEs, see . The method is explicit and fulfills the Error Stepper concept. Step size control is provided but continuous output is available which make this method favourable for many applications.

This class derives from explicit_error_stepper_fsal_base and inherits its interface via CRTP (current recurring template pattern). The method possesses the FSAL (first-same-as-last) property. See explicit_error_stepper_fsal_base for more details.

Template Parameters

1. \texttt{typename State}

The state type.

2. \texttt{typename Value = double}

The value type.

3. \texttt{typename Deriv = State}

The type representing the time derivative of the state.

4. \texttt{typename Time = Value}

The time representing the independent variable - the time.

5. \texttt{typename Algebra = range_algebra}

The algebra type.
6. **typename Operations = default_operations**

The operations type.

7. **typename Resizer = initially_resizer**

The resizer policy type.

### runge_kutta_dopri5 public construct/copy/destroy

1. ```cpp
   runge_kutta_dopri5(const algebra_type & algebra = algebra_type());
```

Constructs the runge_kutta_dopri5 class. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters:
- **algebra**: A copy of algebra is made and stored inside `explicit_stepper_base`.

### runge_kutta_dopri5 public member functions

1. ```cpp
   template<typename System, typename StateIn, typename DerivIn, 
            typename StateOut, typename DerivOut>
   void do_step_impl(System system, const StateIn & in, 
                      const DerivIn & dxdt_in, time_type t, StateOut & out, 
                      DerivOut & dxdt_out, time_type dt);
```

This method performs one step. The derivative $dxdt_{in}$ of $in$ at the time $t$ is passed to the method. The result is updated out-of-place, hence the input is in $in$ and the output in out. Furthermore, the derivative is update out-of-place, hence the input is assumed to be in $dxdt_{in}$ and the output in $dxdt_{out}$. Access to this step functionality is provided by `explicit_error_stepper_fsal_base` and `do_step_impl` should not be called directly.

Parameters:
- **dt**: The step size.
- **dxdt_{in}**: The derivative of $x$ at $t$. $dxdt_{in}$ is not modified by this method.
- **dxdt_{out}**: The result of the new derivative at time $t+dt$.
- **in**: The state of the ODE which should be solved. $in$ is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

2. ```cpp
   template<typename System, typename StateIn, typename DerivIn, 
            typename StateOut, typename DerivOut, typename Err>
   void do_step_impl(System system, const StateIn & in, 
                      const DerivIn & dxdt_in, time_type t, StateOut & out, 
                      DerivOut & dxdt_out, time_type dt, Err & xerr);
```

This method performs one step. The derivative $dxdt_{in}$ of $in$ at the time $t$ is passed to the method. The result is updated out-of-place, hence the input is in $in$ and the output in out. Furthermore, the derivative is update out-of-place, hence the input is assumed to be in $dxdt_{in}$ and the output in $dxdt_{out}$. Access to this step functionality is provided by `explicit_error_stepper_fsal_base` and `do_step_impl` should not be called directly. An estimation of the error is calculated.

Parameters:
- **dt**: The step size.
- **dxdt_{in}**: The derivative of $x$ at $t$. $dxdt_{in}$ is not modified by this method.
- **dxdt_{out}**: The result of the new derivative at time $t+dt$.
- **in**: The state of the ODE which should be solved. $in$ is not modified in this method.
- **out**: The result of the step is written in out.
system The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
t The value of the time, at which the step should be performed.
xerr An estimation of the error.

3. template<typename StateOut, typename StateIn1, typename DerivIn1, typename StateIn2, typename DerivIn2>
   void calc_state(time_type t, StateOut & x, const StateIn1 & x_old, const DerivIn1 & deriv_old, time_type t_old, const StateIn2 &, const DerivIn2 & deriv_new, time_type t_new) const;

This method is used for continuous output and it calculates the state $x$ at a time $t$ from the knowledge of two states $\text{old}_\text{state}$ and $\text{current}_\text{state}$ at time points $t_{\text{old}}$ and $t_{\text{new}}$. It also uses internal variables to calculate the result. Hence this method must be called after two successful do_step calls.

4. template<typename StateIn> void adjust_size(const StateIn & x);

Adjust the size of all temporaries in the stepper manually.

Parameters: $x$ A state from which the size of the temporaries to be resized is deduced.

5. order_type order(void) const;

Returns: Returns the order of the stepper if it used without error estimation.

6. order_type stepper_order(void) const;

Returns: Returns the order of a step if the stepper is used without error estimation.

7. order_type error_order(void) const;

Returns: Returns the order of an error step if the stepper is used without error estimation.

8. template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);

This method performs one step. It transforms the result in-place.

Note

This method uses the internal state of the stepper.

Parameters: $dt$ The step size.

9. template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.
The method performs one step with the stepper passed by Stepper. Additionally to the other methods the derivative of \( x \) is also passed to this method. Therefore, \( \text{dxdt} \) must be evaluated initially:

```cpp
do_step(\system, x, \text{dxdt}, t, dt);
```

Note

This method does NOT use the initial state, since the first derivative is explicitly passed to this method.

The result is updated in place in \( x \) as well as the derivative \( \text{dxdt} \). This method is disabled if Time and StateInOut are of the same type. In this case the method could not be distinguished from other \text{do_step} versions.

Note

This method does not solve the forwarding problem.

Parameters:
- \( \text{dt} \) The step size.
- \( \text{dxdt} \) The derivative of \( x \) at \( t \). After calling \text{do_step} \( \text{dxdt} \) is updated to the new value.
- \( \system \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \) The value of the time, at which the step should be performed.
- \( x \) The state of the ODE which should be solved. After calling \text{do_step} the result is updated in \( x \).

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other \text{do_step} variants.

Note

This method uses the internal state of the stepper.

This method does not solve the forwarding problem.

Parameters:
- \( \text{dt} \) The step size.
- \( \text{in} \) The state of the ODE which should be solved. \( \text{in} \) is not modified in this method.
- \( \text{out} \) The result of the step is written in \text{out}.
- \( \system \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \) The value of the time, at which the step should be performed.
The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and updated by the stepper to its new value at t+dt.

**Note**

This method does not solve the forwarding problem.

This method does NOT use the internal state of the stepper.

Parameters:
- \( dt \) The step size.
- \( dxdt\_in \) The derivative of x at t.
- \( dxdt\_out \) The updated derivative of out at t+dt.
- \( in \) The state of the ODE which should be solved. in is not modified in this method
- \( out \) The result of the step is written in out.
- \( system \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \) The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

**Note**

This method uses the internal state of the stepper.

Parameters:
- \( dt \) The step size.
- \( system \) The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- \( t \) The value of the time, at which the step should be performed.
- \( x \) The state of the ODE which should be solved. x is updated by this method.
- \( xerr \) The estimation of the error is stored in xerr.

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method and updated by this method.
This method does NOT use the internal state of the stepper.

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions. This method is disabled if StateInOut and Time are of the same type.

Note
This method does NOT use the internal state of the stepper.
This method does not solve the forwarding problem.

### Parameters:
- **dt**: The step size.
- **dxdt**: The derivative of x at t. After calling do_step this value is updated to the new value at t+dt.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. After calling do_step the result is updated in x.
- **xerr**: The error estimate is stored in xerr.

```cpp
template<typename System, typename StateIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, time_type t, StateOut & out,
             time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

Note
This method uses the internal state of the stepper.
This method does not solve the forwarding problem.

### Parameters:
- **dt**: The step size.
- **in**: The state of the ODE which should be solved. in is not modified in this method
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **xerr**: The error estimate.

```cpp
template<typename System, typename StateIn, typename DerivIn,
          typename StateOut, typename DerivOut, typename Err>
void do_step(System system, const StateIn & in, const DerivIn & dxdt_in,
             time_type t, StateOut & out, DerivOut & dxdt_out,
             time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated.
This method does NOT use the internal state of the stepper.

This method does not solve the forwarding problem.

**Parameters:**
- **dt** The step size.
- **dxdt\_in** The derivative of x at t.
- **dxdt\_out** The new derivative at \(t+dt\) is written into this variable.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

```cpp
void reset(void);
```

Resets the internal state of this stepper. After calling this method it is safe to use all `do_step` method without explicitly initializing the stepper.

```cpp
template<typename DerivIn> void initialize(const DerivIn & deriv);
```

Initializes the internal state of the stepper.

**Parameters:**
- **deriv** The derivative of x. The next call of `do_step` expects that the derivative of x passed to `do_step` has the value of deriv.

```cpp
template<typename System, typename StateIn>
void initialize(System system, const StateIn & x, time_type t);
```

Initializes the internal state of the stepper.

This method is equivalent to

```cpp
Deriv dxdt;
system( x, dxdt , t );
stepper.initialize( dxdt );
```

**Parameters:**
- **system** The system function for the next calls of `do_step`.
- **t** The current time of the ODE.
- **x** The current state of the ODE.

```cpp
bool is_initialized(void) const;
```

Returns if the stepper is already initialized. If the stepper is not initialized, the first call of `do_step` will initialize the state of the stepper. If the stepper is already initialized the system function can not be safely exchanged between consecutive `do_step` calls.

```cpp
algebra_type & algebra();
```

Returns: A reference to the algebra which is held by this class.

```cpp
const algebra_type & algebra() const;
```
Returns: A const reference to the algebra which is held by this class.

**runge_kutta_dopri5 private member functions**

1. `template<typename StateIn> bool resize_k_x_tmp_impl(const StateIn & x);`

2. `template<typename StateIn> bool resize_dxdt_tmp_impl(const StateIn & x);`

**Header `<boost/numeric/odeint/stepper/runge_kutta_fehlberg78.hpp>`**

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename State, typename Value = double,
                     typename Deriv = State, typename Time = Value,
                     typename Algebra = range_algebra,
                     typename Operations = default_operations,
                     typename Resizer = initially_resizer>
            class runge_kutta_fehlberg78;
        }
    }
}
```

**Class template runge_kutta_fehlberg78**

`boost::numeric::odeint::runge_kutta_fehlberg78` — The Runge-Kutta Fehlberg 78 method.
Synopsis

// In header: <boost/numeric/odeint/stepper/runge_kutta_fehlberg78.hpp>

template<typename State, typename Value = double, typename Deriv = State,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class runge_kutta_fehlberg78 : public boost::numeric::odeint::explicit_error_generic_rk< Stage,
   Count, Order, StepperOrder, ErrorOrder, State, Value, Deriv, Time, Algebra, Operations, Resizer >
{
  public:
    // types
    typedef stepper_base_type::state_type state_type;
    typedef stepper_base_type::value_type value_type;
    typedef stepper_base_type::deriv_type deriv_type;
    typedef stepper_base_type::time_type time_type;
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::operations_type operations_type;
    typedef stepper_base_type::resizer_type resizer_type;

    // construct/copy/destruct
    runge_kutta_fehlberg78(const algebra_type & = algebra_type());

    // public member functions
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut, typename Err>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                       StateOut &, time_type, Err &);
    template<typename System, typename StateIn, typename DerivIn,
             typename StateOut>
    void do_step_impl(System, const StateIn &, const DerivIn &, time_type,
                       StateOut &, time_type);
    template<typename System, typename StateIn> void adjust_size(const StateIn &);
    order_type order(void) const;
    order_type stepper_order(void) const;
    order_type error_order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut, typename DerivIn>
    boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
    do_step(System, StateInOut &, const DerivIn &, time_type, time_type, time_type);
    template<typename System, typename StateIn, typename StateOut, typename Err>
    void do_step(System, StateIn &, const DerivIn &, time_type, StateOut &,
                 time_type, Err &);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, StateOut &,
                 time_type, time_type);
    template<typename System, typename StateOut, typename Err>
    void do_step(System, StateOut &, time_type, time_type, time_type, Err &);
    template<typename System, typename StateOut, typename DerivIn,
             typename Err>
    boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
    do_step(System, StateOut &, const DerivIn &, time_type, time_type, time_type,
             time_type, Err &);
void do_step(System, const StateIn &, time_type, StateOut &, time_type, Err &);

template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err>
void do_step(System, const StateIn &, const DerivIn &, time_type, StateOut &, time_type, Err &);

algebra_type & algebra();
const algebra_type & algebra() const;

Description

The Runge-Kutta Fehlberg 78 method is a standard method for high-precision applications. The method is explicit and fulfills the Error Stepper concept. Step size control is provided but continuous output is not available for this method.

This class derives from explicit_error_stepper_base and inherits its interface via CRTP (current recurring template pattern). Furthermore, it derivs from explicit_error_generic_rk which is a generic Runge-Kutta algorithm with error estimation. For more details see explicit_error_stepper_base and explicit_error_generic_rk.

Template Parameters

1. *typename State*
   
   The state type.

2. *typename Value = double*
   
   The value type.

3. *typename Deriv = State*
   
   The type representing the time derivative of the state.

4. *typename Time = Value*
   
   The time representing the independent variable - the time.

5. *typename Algebra = range_algebra*
   
   The algebra type.

6. *typename Operations = default_operations*
   
   The operations type.

7. *typename Resizer = initially_resizer*
   
   The resizer policy type.

runge_kutta_fehlberg78 public construct/copy/destruct

1. runge_kutta_fehlberg78(const algebra_type & algebra = algebra_type());
Constructs the `runge_kutta_cash_fehlberg78` class. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters: `algebra` A copy of `algebra` is made and stored inside `explicit_stepper_base`.

### `runge_kutta_fehlberg78` public member functions

1. ```
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut, typename Err>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                      time_type t, StateOut & out, time_type dt, Err & xerr);
```  
   This method performs one step. The derivative `dxdt` of `in` at the time `t` is passed to the method. The result is updated out-of-place, hence the input is in `in` and the output in `out`. Furthermore, an estimation of the error is stored in `xerr`. `do_step_impl` is used by `explicit_error_stepper_base`.

   Parameters:  
   - `dt` The step size.  
   - `dxdt` The derivative of `x` at `t`.  
   - `in` The state of the ODE which should be solved. `in` is not modified in this method  
   - `out` The result of the step is written in `out`.  
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
   - `t` The value of the time, at which the step should be performed.  
   - `xerr` The result of the error estimation is written in `xerr`.

2. ```
   template<typename System, typename StateIn, typename DerivIn,
            typename StateOut>
   void do_step_impl(System system, const StateIn & in, const DerivIn & dxdt,
                      time_type t, StateOut & out, time_type dt);
```  
   This method performs one step. The derivative `dxdt` of `in` at the time `t` is passed to the method. The result is updated out-of-place, hence the input is in `in` and the output in `out`. Access to this step functionality is provided by `explicit_stepper_base` and `do_step_impl` should not be called directly.

   Parameters:  
   - `dt` The step size.  
   - `dxdt` The derivative of `x` at `t`.  
   - `in` The state of the ODE which should be solved. `in` is not modified in this method  
   - `out` The result of the step is written in `out`.  
   - `system` The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.  
   - `t` The value of the time, at which the step should be performed.

3. ```
   template<typename StateIn> void adjust_size(const StateIn & x);
```  
   Adjust the size of all temporaries in the stepper manually.

   Parameters:  
   - `x` A state from which the size of the temporaries to be resized is deduced.

4. ```
   order_type order(void) const;
```  
   Returns: Returns the order of the stepper if it used without error estimation.

5. ```
   order_type stepper_order(void) const;
```  
   Returns: Returns the order of a step if the stepper is used without error estimation.

6. ```
   order_type error_order(void) const;
```
Returns: Returns the order of an error step if the stepper is used without error estimation.

7. template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & x, time_type t, time_type dt);

This method performs one step. It transforms the result in-place.

Parameters:
- **dt**  The step size.
- **system**  The system function to solve, hence the r.h.s. of the ordinary differential equation. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.
- **x**  The state of the ODE which should be solved. After calling do_step the result is updated in x.

8. template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & x, time_type t, time_type dt);

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

9. template<typename System, typename StateInOut, typename DerivIn>
   boost::disable_if< boost::is_same< DerivIn, time_type >, void >::type
   do_step(System system, StateInOut & x, const DerivIn & dxdt, time_type t, time_type dt);

The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```cpp
sys( x , dxdt , t );
stepper.do_step( sys , x , dxdt , t , dt );
```

The result is updated in place in x. This method is disabled if Time and Deriv are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt**  The step size.
- **dxdt**  The derivative of x at t.
- **system**  The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**  The value of the time, at which the step should be performed.
- **x**  The state of the ODE which should be solved. After calling do_step the result is updated in x.

10. template<typename System, typename StateIn, typename StateOut>
    boost::disable_if< boost::is_same< StateIn, time_type >, void >::type
    do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. This method is disabled if StateIn and Time are the same type. In this case the method can not be distinguished from other do_step variants.

**Note**

This method does not solve the forwarding problem.
Parameters:

- **dt**: The step size.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper. It is supposed to be used in the following way:

```cpp
sys( in , dxdt , t );
stepper.do_step( sys , in , dxdt , t , out , dt );
```

This method is disabled if DerivIn and Time are of same type.

**Note**

This method does not solve the forwarding problem.

Parameters:

- **dt**: The step size.
- **dxdt**: The derivative of x at t.
- **in**: The state of the ODE which should be solved. in is not modified in this method.
- **out**: The result of the step is written in out.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.

The method performs one step with the stepper passed by Stepper and estimates the error. The state of the ODE is updated in-place.

Parameters:

- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **x**: The state of the ODE which should be solved. x is updated by this method.
- **xerr**: The estimation of the error is stored in xerr.

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.

Parameters:

- **dt**: The step size.
- **system**: The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t**: The value of the time, at which the step should be performed.
- **dxdt**: The derivative of x at t.
- **xerr**: The estimation of the error is stored in xerr.

Second version to solve the forwarding problem, can be called with Boost.Range as StateInOut.
The method performs one step with the stepper passed by Stepper. Additionally to the other method the derivative of x is also passed to this method. It is supposed to be used in the following way:

```
sys( x, dxdt, t );
stepper.do_step( sys, x, dxdt, t, dt, xerr );
```

The result is updated in place in x. This method is disabled if Time and DerivIn are of the same type. In this case the method could not be distinguished from other do_step versions.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **x** The state of the ODE which should be solved. After calling do_step the result is updated in x.
- **xerr** The error estimate is stored in x.

```
template<typename System, typename StateIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the error is estimated.

**Note**

This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

```
template<typename System, typename StateIn, typename DerivIn, typename StateOut, typename Err>
void do_step(System system, const StateIn & in, const DerivIn & dxdt, time_type t, StateOut & out, time_type dt, Err & xerr);
```

The method performs one step with the stepper passed by Stepper. The state of the ODE is updated out-of-place. Furthermore, the derivative of x at t is passed to the stepper and the error is estimated. It is supposed to be used in the following way:

```
sys( in, dxdt, t );
stepper.do_step( sys, in, dxdt, t, out, dt );
```

This method is disabled if DerivIn and Time are of same type.
Note

This method does not solve the forwarding problem.

Parameters:
- **dt** The step size.
- **dxdt** The derivative of x at t.
- **in** The state of the ODE which should be solved. in is not modified in this method.
- **out** The result of the step is written in out.
- **system** The system function to solve, hence the r.h.s. of the ODE. It must fulfill the Simple System concept.
- **t** The value of the time, at which the step should be performed.
- **xerr** The error estimate.

```cpp
boost::numeric::odeint::stepper_tag
```

Returns: A reference to the algebra which is held by this class.

```cpp
const boost::numeric::odeint::stepper_tag
```

Returns: A const reference to the algebra which is held by this class.

Header `<boost/numeric/odeint/stepper/stepper_categories.hpp>`
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct stepper_tag {
};
```

**Struct error_stepper_tag**

boost::numeric::odeint::error_stepper_tag

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct error_stepper_tag : public boost::numeric::odeint::stepper_tag {
};
```

**Struct explicit_error_stepper_tag**

boost::numeric::odeint::explicit_error_stepper_tag

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct explicit_error_stepper_tag :
    public boost::numeric::odeint::error_stepper_tag {
};
```

**Struct explicit_error_stepper_fsal_tag**

boost::numeric::odeint::explicit_error_stepper_fsal_tag

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct explicit_error_stepper_fsal_tag :
    public boost::numeric::odeint::error_stepper_tag {
};
```

**Struct controlled_stepper_tag**

boost::numeric::odeint::controlled_stepper_tag

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct controlled_stepper_tag {
};
```
Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct controlled_stepper_tag {
};

Struct explicit_controlled_stepper_tag
boost::numeric::odeint::explicit_controlled_stepper_tag

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct explicit_controlled_stepper_tag :
  public boost::numeric::odeint::controlled_stepper_tag
{
};

Struct explicit_controlled_stepper_fsal_tag
boost::numeric::odeint::explicit_controlled_stepper_fsal_tag

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct explicit_controlled_stepper_fsal_tag :
  public boost::numeric::odeint::controlled_stepper_tag
{
};

Struct dense_output_stepper_tag
boost::numeric::odeint::dense_output_stepper_tag

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct dense_output_stepper_tag {
};

Struct template base_tag
boost::numeric::odeint::base_tag
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

template<typename tag>
struct base_tag {
};
```

Struct `base_tag<stepper_tag>`

```cpp
boost::numeric::odeint::base_tag<stepper_tag>
```

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<stepper_tag> {
    // types
    typedef stepper_tag type;
};
```

Struct `base_tag<error_stepper_tag>`

```cpp
boost::numeric::odeint::base_tag<error_stepper_tag>
```

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<error_stepper_tag> {
    // types
    typedef stepper_tag type;
};
```

Struct `base_tag<explicit_error_stepper_tag>`

```cpp
boost::numeric::odeint::base_tag<explicit_error_stepper_tag>
```

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<explicit_error_stepper_tag> {
    // types
    typedef stepper_tag type;
};
```
 Struct base_tag<explicit_error_stepper_fsal_tag>
boost::numeric::odeint::base_tag<explicit_error_stepper_fsal_tag>

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<explicit_error_stepper_fsal_tag> {  
    // types
    typedef stepper_tag type;
};

 Struct base_tag<controlled_stepper_tag>
boost::numeric::odeint::base_tag<controlled_stepper_tag>

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<controlled_stepper_tag> {  
    // types
    typedef controlled_stepper_tag type;
};

 Struct base_tag<explicit_controlled_stepper_tag>
boost::numeric::odeint::base_tag<explicit_controlled_stepper_tag>

Synopsis

// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<explicit_controlled_stepper_tag> {  
    // types
    typedef controlled_stepper_tag type;
};

 Struct base_tag<explicit_controlled_stepper_fsal_tag>
boost::numeric::odeint::base_tag<explicit_controlled_stepper_fsal_tag>
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<explicit_controlled_stepper_fsal_tag> {
    // types
    typedef controlled_stepper_tag type;
};
```

Struct base_tagdense_output_stepper_tag>

boost::numeric::odeint::base_tag<dense_output_stepper_tag>

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/stepper_categories.hpp>

struct base_tag<dense_output_stepper_tag> {
    // types
    typedef dense_output_stepper_tag type;
};
```

Header <boost/numeric/odeint/stepper/symplectic_euler.hpp>

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Coor, typename Momentum = Coor, 
                typename Value = double, typename CoorDeriv = Coor, 
                typename MomentumDeriv = Coor, typename Time = Value, 
                typename Algebra = range_algebra, 
                typename Operations = default_operations, 
                typename Resizer = initially_resizer>
                class symplectic_euler;
        }
    }
}
```

Class template symplectic_euler

boost::numeric::odeint::symplectic_euler — Implementation of the symplectic Euler method.
Synopsis

// In header: <boost/numeric/odeint/stepper/symplectic_euler.hpp>

```cpp
template<typename Coor, typename Momentum = Coor, typename Value = double,
         typename CoorDeriv = Coor, typename MomentumDeriv = Coor,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class symplectic_euler : public boost::numeric::odeint::symplectic_nystroem_stepper_base<
    NumOfStages, Order, Coor, Momentum, Value, CoorDeriv, MomentumDeriv, Time, Algebra, Operations, Resizer >
{
public:
    // types
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::value_type value_type;

    // construct/copy/destruct
    symplectic_euler(const algebra_type & = algebra_type());

    // public member functions
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, const StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename CoorInOut, typename MomentumInOut>
    void do_step(System, CoorInOut &, MomentumInOut &, time_type, time_type);
    template<typename System, typename CoorInOut, typename MomentumInOut>
    void do_step(System, const CoorInOut &, const MomentumInOut &, time_type,
                 time_type);
    template<typename System, typename StateIn, typename StateOut>
    void do_step(System, const StateIn &, time_type, StateOut &, time_type);
    template<typename StateType> void adjust_size(const StateType &);
    const coef_type & coef_a(void) const;
    const coef_type & coef_b(void) const;
    algebra_type & algebra();
    const algebra_type & algebra() const;
};
```

Description

The method is of first order and has one stage. It is described HERE.

Template Parameters

1. `typename Coor`  
The type representing the coordinates q.

2. `typename Momentum = Coor`  
The type representing the coordinates p.

3. `typename Value = double`  
The basic value type. Should be something like float, double or a high-precision type.
4. \texttt{typename CoorDeriv = Coor}

The type representing the time derivative of the coordinate \(dq/dt\).

5. \texttt{typename MomentumDeriv = Coor}

6. \texttt{typename Time = Value}

The type representing the time \(t\).

7. \texttt{typename Algebra = range_algebra}

The algebra.

8. \texttt{typename Operations = default_operations}

The operations.

9. \texttt{typename Resizer = initially_resizer}

The resizer policy.

\textbf{symplectic_euler public construct/copy/destruct}

1. \texttt{symplectic_euler(const algebra_type & algebra = algebra_type());}

Constructs the \texttt{symplectic_euler}. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters:
\begin{itemize}
\item \texttt{algebra} A copy of algebra is made and stored inside \texttt{explicit_stepper_base}.
\end{itemize}

\textbf{symplectic_euler public member functions}

1. \texttt{order_type order(void) const;}

Returns: Returns the order of the stepper.

2. \texttt{template<typename System, typename StateInOut>
   void do_step(System system, const StateInOut & state, time_type t,
                time_type dt);}\

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \(dq/dt = p\). The state is updated in-place.

\section*{Note}

boost::ref or std::ref can be used for the system as well as for the state. So, it is correct to write \texttt{step-per.do_step(make_pair(std::ref(fq), std::ref(fp)), make_pair(std::ref(q), std::ref(p)), t, dt)}.

This method solves the forwarding problem.
Parameters:

- \( dt \) The time step.
- \( state \) The state of the ODE. It is a pair of Coor and Momentum. The state is updated in-place, therefore, the new value of the state will be written into this variable.
- \( system \) The system, can be represented as a pair of two function object or one function object. See above.
- \( t \) The time of the ODE. It is not advanced by this method.

3. 

```cpp
template<typename System, typename StateInOut>
void do_step(System system, StateInOut & state, time_type t, time_type dt);
```

Same function as above. It differs only in a different const specifier in order to solve the forwarding problem, can be used with Boost.Range.

4. 

```cpp
template<typename System, typename CoorInOut, typename MomentumInOut>
void do_step(System system, CoorInOut & q, MomentumInOut & p, time_type t, time_type dt);
```

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( dq/dt = p \). The state is updated in-place.

5. 

```cpp
template<typename System, typename CoorInOut, typename MomentumInOut>
void do_step(System system, const CoorInOut & q, const MomentumInOut & p, time_type t, time_type dt);
```

This method solves the forwarding problem.

- \( dt \) The time step.
- \( p \) The momentum of the ODE. It is updated in-place. Therefore, the new value of the momentum will be written into this variable.
- \( q \) The coordinate of the ODE. It is updated in-place. Therefore, the new value of the coordinate will be written into this variable.
- \( system \) The system, can be represented as a pair of two function object or one function object. See above.
- \( t \) The time of the ODE. It is not advanced by this method.

6. 

```cpp
template<typename System, typename StateIn, typename StateOut>
void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);
```

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( dq/dt = p \). The state is updated out-of-place.
Boost::Numeric.Odeint

Note

boost::ref or std::ref can be used for the system. So, it is correct to write stepper.do_step( make_pair( std::ref( fq ), std::ref( fp ) ), x_in, t, x_out, dt ).

This method NOT solve the forwarding problem.

Parameters:

- **dt**  
The time step.
- **in**  
The state of the ODE, which is a pair of coordinate and momentum. The state is updated out-of-place, therefore the new value is written into out
- **out**  
The new state of the ODE.
- **system**  
The system, can be represented as a pair of two function object or one function object. See above.
- **t**  
The time of the ODE. It is not advanced by this method.

7. 

```cpp
template<typename StateType> void adjust_size(const StateType & x);
```

Adjust the size of all temporaries in the stepper manually.

Parameters:

- **x**  
A state from which the size of the temporaries to be resized is deduced.

8. 

```cpp
const coef_type & coef_a(void) const;
```

Returns the coefficients a.

9. 

```cpp
const coef_type & coef_b(void) const;
```

Returns the coefficients b.

10. 

```cpp
algebra_type & algebra();
```

Returns:  
A reference to the algebra which is held by this class.

11. 

```cpp
const algebra_type & algebra() const;
```

Returns:  
A const reference to the algebra which is held by this class.

**Header** `<boost/numeric/odeint/stepper/symplectic_rkn_sb3a_m4_mclachlan.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<typename Coor, typename Momentum = Coor,
                typename Value = double, typename CoorDeriv = Coor,
                typename MomentumDeriv = Coor, typename Time = Value,
                typename Algebra = range_algebra,
                typename Operations = default_operations,
                typename Resizer = initially_resizer>
                class symplectic_rkn_sb3a_m4_mclachlan;
        }
    }
}
```
Class template symplectic_rkn_sb3a_m4_mclachlan

boost::numeric::odeint::symplectic_rkn_sb3a_m4_mclachlan — Implementation of the symmetric B3A Runge-Kutta Nystroem method of fifth order.

Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/symplectic_rkn_sb3a_m4_mclachlan.hpp>

template<typename Coor, typename Momentum = Coor, typename Value = double,
         typename CoorDeriv = Coor, typename MomentumDeriv = Coor,
         typename Time = Value, typename Algebra = range_algebra,
         typename Operations = default_operations,
         typename Resizer = initially_resizer>
class symplectic_rkn_sb3a_m4_mclachlan : public boost::numeric::odeint::symplectic_nystroem_step_per_base< NumOfStages, Order, Coor, Momentum, Value, CoorDeriv, MomentumDeriv, Time, Algebra, Operations, Resizer > 
{
    public:
        // types
        typedef stepper_base_type::algebra_type algebra_type;
        typedef stepper_base_type::value_type value_type;

        // construct/copy/destruct
        symplectic_rkn_sb3a_m4_mclachlan(const algebra_type & = algebra_type());

        // public member functions
        order_type order(void) const;
        template<typename System, typename StateInOut>
        void do_step(System, const StateInOut &, time_type, time_type);
        template<typename System, typename StateInOut>
        void do_step(System, StateInOut &, time_type, time_type);
        template<typename System, typename CoorInOut, typename MomentumInOut>
        void do_step(System, CoorInOut &, MomentumInOut &, time_type, time_type);
        template<typename System, typename CoorInOut, typename MomentumInOut>
        void do_step(System, const CoorInOut &, const MomentumInOut &, time_type,
                     time_type);
        template<typename System, typename StateIn, typename StateOut>
        void do_step(System, const StateIn &, time_type, StateOut &, time_type);
        template<typename StateType> void adjust_size(const StateType &);
        const coef_type & coef_a(void) const;
        const coef_type & coef_b(void) const;
        algebra_type & algebra();
        const algebra_type & algebra() const;
    };
```

Description

The method is of fourth order and has five stages. It is described HERE. This method can be used with multiprecision types since the coefficients are defined analytically.

ToDo: add reference to paper.

Template Parameters

1. **typename Coor**

The type representing the coordinates q.
The type representing the coordinates p.

The basic value type. Should be something like float, double or a high-precision type.

The type representing the time derivative of the coordinate dq/dt.

The type representing the time t.

The algebra.

The operations.

The resizer policy.

Constructs the `symplectic_rkn_sb3a_m4_mclachlan`. This constructor can be used as a default constructor if the algebra has a default constructor.

Parameters: algebra A copy of algebra is made and stored inside `explicit_stepper_base`.

Returns: Returns the order of the stepper.

```cpp
order_type order(void) const;
```

```cpp
template< typename System, typename StateInOut >
void do_step(System system, const StateInOut & state, time_type t, time_type dt);
```
This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( \frac{dq}{dt} = p \). The state is updated in-place.

**Note**

boost::ref or std::ref can be used for the system as well as for the state. So, it is correct to write `stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , make_pair( std::ref( q ) , std::ref( p ) ) , t , dt ).`

This method solves the forwarding problem.

**Parameters:**
- \( dt \) The time step.
- \( state \) The state of the ODE. It is a pair of Coor and Momentum. The state is updated in-place, therefore, the new value of the state will be written into this variable.
- \( system \) The system, can be represented as a pair of two function object or one function object. See above.
- \( t \) The time of the ODE. It is not advanced by this method.

3. ```
   template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & state, time_type t, time_type dt);
```

Same function as above. It differs only in a different const specifier in order to solve the forwarding problem, can be used with Boost.Range.

4. ```
   template<typename System, typename CoorInOut, typename MomentumInOut>
   void do_step(System system, CoorInOut & q, MomentumInOut & p, time_type t, time_type dt);
```

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( \frac{dq}{dt} = p \). The state is updated in-place.

**Note**

boost::ref or std::ref can be used for the system. So, it is correct to write `stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) ) , q , p , t , dt ).`

This method solves the forwarding problem.

**Parameters:**
- \( dt \) The time step.
- \( p \) The momentum of the ODE. It is updated in-place. Therefore, the new value of the momentum will be written info this variable.
- \( q \) The coordinate of the ODE. It is updated in-place. Therefore, the new value of the coordinate will be written into this variable.
- \( system \) The system, can be represented as a pair of two function object or one function object. See above.
- \( t \) The time of the ODE. It is not advanced by this method.

5. ```
   template<typename System, typename CoorInOut, typename MomentumInOut>
   void do_step(System system, const CoorInOut & q, const MomentumInOut & p, time_type t, time_type dt);
```

Same function as `do_step( system , q , p , t , dt ).` It differs only in a different const specifier in order to solve the forwarding problem, can be called with Boost.Range.
template<typename System, typename StateIn, typename StateOut>
void do_step(System system, const StateIn & in, time_type t, StateOut & out,
    time_type dt);

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( dq/dt = p \). The state is updated out-of-place.

**Note**

boost::ref or std::ref can be used for the system. So, it is correct to write `stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , x_in , t , x_out , dt )`.

This method NOT solve the forwarding problem.

Parameters:
- **dt** The time step.
- **in** The state of the ODE, which is a pair of coordinate and momentum. The state is updated out-of-place, therefore the new value is written into out.
- **out** The new state of the ODE.
- **system** The system, can be represented as a pair of two function object or one function object. See above.
- **t** The time of the ODE. It is not advanced by this method.

7. template<typename StateType> void adjust_size(const StateType & x);

Adjust the size of all temporaries in the stepper manually.

Parameters:
- **x** A state from which the size of the temporaries to be resized is deduced.

8. const coef_type & coef_a(void) const;

Returns the coefficients a.

9. const coef_type & coef_b(void) const;

Returns the coefficients b.

10. algebra_type & algebra();

Returns: A reference to the algebra which is held by this class.

11. const algebra_type & algebra() const;

Returns: A const reference to the algebra which is held by this class.
Header `<boost/numeric/odeint/stepper/symplectic_rkn_sb3a_mclachlan.hpp>`

```cpp
namespace boost {
    namespace numeric {
        namespace odeint {
            template<
                typename Coor,
                typename Momentum = Coor,
                typename Value = double,
                typename CoorDeriv = Coor,
                typename MomentumDeriv = Coor,
                typename Time = Value,
                typename Algebra = range_algebra,
                typename Operations = default_operations,
                typename Resizer = initially_resizer>
            class symplectic_rkn_sb3a_mclachlan;
        }
    }
}
```

Class template `symplectic_rkn_sb3a_mclachlan`

Boost::numeric::odeint::symplectic_rkn_sb3a_mclachlan — Implement of the symmetric B3A method of Runge-Kutta-Nystroem method of sixth order.
Synopsis

```cpp
// In header: <boost/numeric/odeint/stepper/symplectic_rkn_sb3a_mclachlan.hpp>

template<typename Coor, typename Momentum = Coor, typename Value = double,
          typename CoorDeriv = Coor, typename MomentumDeriv = Coor,
          typename Time = Value, typename Algebra = range_algebra,
          typename Operations = default_operations,
          typename Resizer = initially_resizer>
class symplectic_rkn_sb3a_mclachlan : public boost::numeric::odeint::symplectic_nystroem_step<
          per_base< NumOfStages, Order, Coor, Momentum, Value, CoorDeriv, MomentumDeriv, Time, Algebra, Operations, Resizer >
        
public:
    // types
    typedef stepper_base_type::algebra_type algebra_type;
    typedef stepper_base_type::value_type value_type;

    // construct/copy/destroy
    symplectic_rkn_sb3a_mclachlan(const algebra_type & = algebra_type());

    // public member functions
    order_type order(void) const;
    template<typename System, typename StateInOut>
    void do_step(System, const StateInOut &, time_type, time_type);
    template<typename System, typename StateInOut>
    void do_step(System, StateInOut &, time_type, time_type);
    template<typename System, typename CoorInOut, typename MomentumInOut>
    void do_step(System, CoorInOut &, MomentumInOut &, time_type, time_type);   
    template<typename System, typename CoorInOut, typename MomentumInOut, typename StateIn, typename StateOut>
    void do_step(System, const CoorInOut &, const MomentumInOut &, time_type, StateOut &, time_type);
    template<typename CoorType> void adjust_size(const CoorType &);
    const coef_type & coef_a(void) const;
    const coef_type & coef_b(void) const;
    algebra_type & algebra();
    const algebra_type & algebra() const;
};
```

Description

The method is of fourth order and has six stages. It is described HERE. This method cannot be used with multiprecision types since the coefficients are not defined analytically.

ToDo Add reference to the paper.

Template Parameters

1. `typename Coor`
   - The type representing the coordinates q.

2. `typename Momentum = Coor`
   - The type representing the coordinates p.
3. `typename Value = double`  
The basic value type. Should be something like float, double or a high-precision type.

4. `typename CoorDeriv = Coor`  
The type representing the time derivative of the coordinate dq/dt.

5. `typename MomentumDeriv = Coor`  

6. `typename Time = Value`  
The type representing the time t.

7. `typename Algebra = range_algebra`  
The algebra.

8. `typename Operations = default_operations`  
The operations.

9. `typename Resizer = initially_resizer`  
The resizer policy.

**symplectic_rkn_sb3a_mclachlan public construct/copy/destroy**

1. `symplectic_rkn_sb3a_mclachlan(const algebra_type & algebra = algebra_type());`  
Constructs the `symplectic_rkn_sb3a_mclachlan`. This constructor can be used as a default constructor if the algebra has a default constructor.

   Parameters:  
   - `algebra`: A copy of algebra is made and stored inside `explicit_stepper_base`.

**symplectic_rkn_sb3a_mclachlan public member functions**

1. `order_type order(void) const;`  
Returns: Returns the order of the stepper.

2. `template<typename System, typename StateInOut>`  
   
   `void do_step(System system, const StateInOut & state, time_type t, time_type dt);`  
   
   This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial dq/dt = p. The state is updated in-place.
Note

boost::ref or std::ref can be used for the system as well as for the state. So, it is correct to write

```cpp
stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , make_pair( std::ref( q ) , std::ref( p ) ) , t , dt ) .
```

This method solves the forwarding problem.

Parameters:
- **dt** The time step.
- **state** The state of the ODE. It is a pair of Coor and Momentum. The state is updated in-place, therefore, the new value of the state will be written into this variable.
- **system** The system, can be represented as a pair of two function object or one function object. See above.
- **t** The time of the ODE. It is not advanced by this method.

3. ```cpp
   template<typename System, typename StateInOut>
   void do_step(System system, StateInOut & state, time_type t, time_type dt);
```

Same function as above. It differs only in a different const specifier in order to solve the forwarding problem, can be used with Boost.Range.

4. ```cpp
   template<typename System, typename CoorInOut, typename MomentumInOut>
   void do_step(System system, CoorInOut & q, MomentumInOut & p, time_type t, time_type dt);
```

This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( dq/dt = p \). The state is updated in-place.

Note

boost::ref or std::ref can be used for the system. So, it is correct to write

```cpp
stepper.do_step( make_pair( std::ref( fq ) , std::ref( fp ) ) , q , p , t , dt ).
```

This method solves the forwarding problem.

Parameters:
- **dt** The time step.
- **p** The momentum of the ODE. It is updated in-place. Therefore, the new value of the momentum will be written into this variable.
- **q** The coordinate of the ODE. It is updated in-place. Therefore, the new value of the coordinate will be written into this variable.
- **system** The system, can be represented as a pair of two function object or one function object. See above.
- **t** The time of the ODE. It is not advanced by this method.

5. ```cpp
   template<typename System, typename CoorInOut, typename MomentumInOut>
   void do_step(System system, const CoorInOut & q, const MomentumInOut & p, time_type t, time_type dt);
```

Same function as do_step( system , q , p , t , dt ). It differs only in a different const specifier in order to solve the forwarding problem, can be called with Boost.Range.

6. ```cpp
   template<typename System, typename StateIn, typename StateOut>
   void do_step(System system, const StateIn & in, time_type t, StateOut & out, time_type dt);
```
This method performs one step. The system can be either a pair of two function object describing the momentum part and the coordinate part or one function object describing only the momentum part. In this case the coordinate is assumed to be trivial \( \frac{dq}{dt} = p \). The state is updated out-of-place.

**Note**

boost::ref or std::ref can be used for the system. So, it is correct to write `stepper.do_step( make_pair( std::ref( fq ), std::ref( fp ) ), x_in, t, x_out, dt )`.

This method **NOT** solve the forwarding problem.

**Parameters:**
- **dt** The time step.
- **in** The state of the ODE, which is a pair of coordinate and momentum. The state is updated out-of-place, therefore the new value is written into out.
- **out** The new state of the ODE.
- **system** The system, can be represented as a pair of two function object or one function object. See above.
- **t** The time of the ODE. It is not advanced by this method.

7. `template<typename StateType> void adjust_size(const StateType & x);`

Adjust the size of all temporaries in the stepper manually.

**Parameters:**
- **x** A state from which the size of the temporaries to be resized is deduced.

8. `const coef_type & coef_a(void) const;`

Returns the coefficients a.

9. `const coef_type & coef_b(void) const;`

Returns the coefficients b.

10. `algebra_type & algebra();`

Returns: A reference to the algebra which is held by this class.

11. `const algebra_type & algebra() const;`

Returns: A const reference to the algebra which is held by this class.
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